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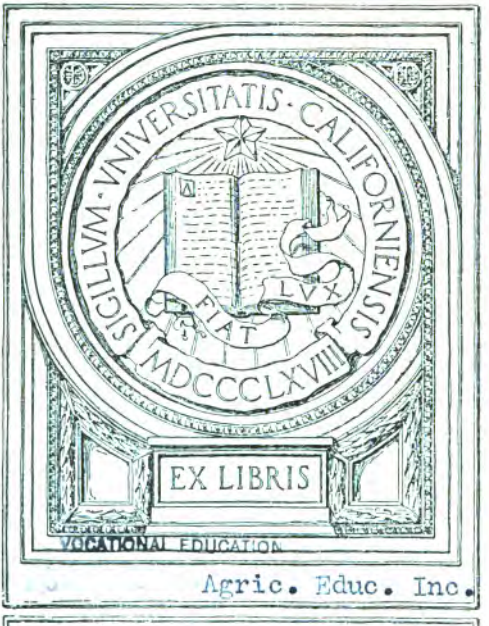

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MATHEMATICS

FOR

AGRICULTURAL STUDENTS

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MATHEMATICS

FOR

AGRICULTURAL STUDENTS

BY

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ASSISTANT PROFESSOR OF MATHEMATICS

UNIVERSITY OF WISCONSIN

FIRST EDITION

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PREFACE

The present book was designed as the text for a working course in elementary Mathematics given in the College of Agriculture of the University of Wisconsin. It is not a text book of Advanced Algebra, Trigonometry, Analytic Geometry, or "Practical Mathematics." Students pursuing a scientific course, in which but a year of college Mathematics is offered, receive little profit from a formal course in Advanced Algebra, Trigonometry, or Analytic Geometry. On the other hand, such students benefit little from a so-called "practical" course of a type which does little more than train them to substitute in given formulas, or to use formulas merely as a means to an end. Such a course does little to develop the habit or power of clear and logical thinking.

This book is the outgrowth of mimeograph notes used for the past three or four years with undergraduate students of agriculture. With the exception of the illustrations and exercises, the book contains, nevertheless, nothing which is of exclusive interest to agricultural students. It may be used equally well with any class of scientific students who desire only a short course in mathematics beyond elementary Algebra and Geometry.

The plan of the book is:

First, to select material primarily on the basis of its usefulness to scientific students.

Second, to illustrate the principles with problems of interest to the agricultural student, or with problems having direct application to his work.

Third, to give minute and detailed explanations of all new work, *and to assume the minimum attainments in mathematical preparation on the part of the student.* On the other hand, details are often omitted from work that is not new to the student, in order that they may be supplied by the student himself.

Fourth, to give detailed directions for doing work "at home" so

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that the student of moderate mathematical attainments may, nevertheless, "learn by doing."

Fifth, to include certain material not elsewhere readily accessible to the student of science, for the purpose of rendering the book useful for reference throughout the four years of the college course.

Sixth, to select the topics and the amount of material under each topic so that either a half year or a full year may be devoted to the work.

In writing this book the author had in mind the preparation of a text which would:

First, train the student to do neat and careful work.

Second, encourage the student to make further use of elementary Algebra and Geometry.

Third, develop in the student the habit of careful and logical thinking.

Fourth, train the student to study a problem with a view of discovering the shortest and easiest method of handling it, rather than attacking it by the "first thought-of" method.

Fifth, show the student, by illustrations and exercises, how mathematics may be helpful in pursuing other subjects of study and useful in a "practical" way.

While this book includes enough material for a year's course of study, it is believed that it does not contain too much material for a shorter course. The inclusion of additional material in the text, will at least let the student know of the existence of subjects in mathematics not covered by him in the class room. Even this superficial knowledge may be very helpful to him later in connection with other scientific work, especially if he knows he can find brief discussions in a familiar book.

The author takes this opportunity to thank Professor C. S. Slichter for assistance in the preparation of the entire manuscript; and to acknowledge his indebtedness to Professor E. V. Huntington for valuable suggestions and criticisms. Acknowledgments are due Mr. E. Taylor and Mr. T. C. Fry for suggestions based upon their use of the preliminary notes in the class room.

A very brief review of elementary Algebra is given in the introduction, besides a list of materials and instruments. In the

appendix is given, for reference, a list of common mathematical symbols and a few formulas of mensuration.

Material in the book which may be omitted when less than a full year is devoted to the course, is indicated by enclosing exercise and section numbers, and chapter headings within brackets.

HENRY C. WOLFF.

UNIVERSITY OF WISCONSIN
September, 18, 1914

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MATHEMATICS FOR AGRICULTURAL STUDENTS

INTRODUCTION

1. Instruments and Materials. A large number of problems in this book are to be solved graphically or solved both graphically and analytically, and a large number of the exercises are exercises in graphic representation of observed or computed data. In order that this work may be performed neatly and accurately, it is necessary that the student have at least a few simple drawing instruments. The indispensable instruments, together with other material required for the work, are here listed and briefly described.

1. **Two 4H Drawing Pencils.** One, sharpened to a fine point, is used for marking points upon paper, or for sketching free-hand. The other, sharpened to a chisel-point, is used for drawing straight lines. Two views of a chisel-pointed pencil are given in Fig. 1. When drawing lines, a flat side of the chisel-point is held against the straight edge of the guide, thus causing the pencil always to move in a direction with the chisel edge and to produce a fine and distinct line. For fine and accurate work, both pencil points must be kept sharp. Neither point ought to become so rounded or flat that any appreciable metallic luster will appear at the wearing surface of the graphite. A good way of keeping pencil points sharp is by occasionally whetting upon fine

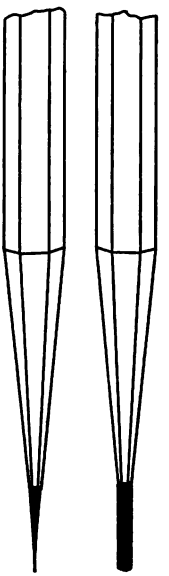


FIG. 1.—Two views of a chisel-pointed pencil.

sand-paper. Small pads of sand-paper for this purpose may be procured from stationers or dealers in drawing materials. Hexagonal pencils should always be used. The student may prefer to have one pencil sharpened at both ends; one end with a round point, and the other end with a chisel-point.

2. **One 3H Pencil,** sharpened to a round point for lettering.

3. **A small drawing board** made of soft wood to which the drawing paper may be fastened with thumb-tacks. A drawing board should have at least one straight edge along which the T-square may slide. A board 12 by 18 inches will be large enough.

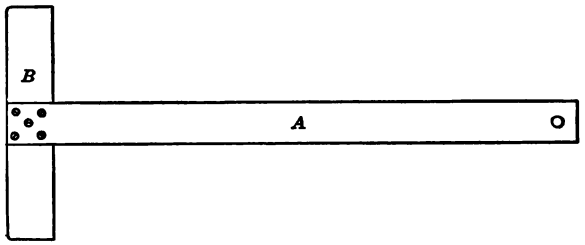


FIG. 2.—T-square.

4. **A small T-square** about 15 inches long. A T-square, Fig. 2, consists simply of a thin, straight-edged blade, *A*, screwed to a heavier block of wood, the head, *B*. The inside of the head has a smooth, straight edge. The T-square is placed upon the drawing board with its head held against the left-hand edge of the drawing board. By sliding it up and down with the left hand, the draughtsman is enabled to draw with it as guide a series of parallel lines running from left to right. From this use of the T-square, it is seen that the edge of the drawing board along which it slides must be smooth and straight, otherwise the lines drawn along the blade of the square would not be parallel. It is not at all necessary that the blade of a T-square be exactly at right angles to the inner edge of the head.

5. **A 60° and a 45° Transparent Celluloid Triangle.** Drawing triangles are thin triangular guides, usually made of wood or celluloid, one angle of which is a right angle. (See Fig. 3.)

In a 60° triangle, the acute angles are 60° and 30°, and in a 45° triangle, the two acute angles are each 45°.

With these triangles straight lines may be drawn making angles of 15°, 30°, 45°, 60°, 75°, and 90° with the lines drawn with the T-square as a guide.

For the work in this book 6-inch triangles will be found large enough, although an additional larger 60° triangle will be found very convenient.

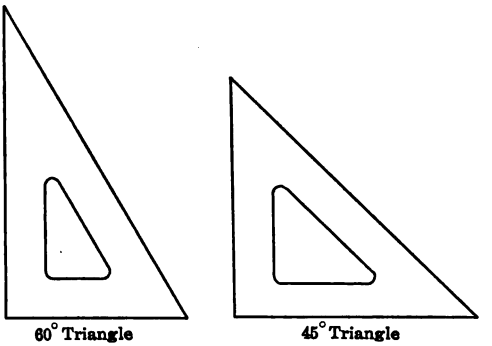


FIG. 3.—60° and 45° triangles.

6. **A protractor** is an instrument for laying off or measuring angles upon a drawing. One, consisting of angular graduations upon a semicircular piece of transparent celluloid, about 6 inches in diameter, is recommended for this work. (See Fig. 4.)

7. **A scale** divided decimally (the unit of length divided into tenths) should be used rather than one in which the unit is divided into eighths, twelfths, or sixteenths. It is advised that a triangular box-wood scale about 4 inches long, or longer, be used in this work. (See Fig. 5.)

8. A pair of 6-inch pencil **compasses** used for drawing circles and arcs of circles. The lead in a cheap pair of compasses usually is not found satisfactory and should be replaced by a good piece of medium hardness, sharpened to a narrow chisel-point. The lead should be so adjusted that when the two legs of the compasses are brought together the two points will be together or directly opposite each other.

9. A good rubber **eraser** for erasing lead pencil marks.

10. Twelve small thumb-tacks.

11. Fifty sheets of squared paper, form *M*1, used for plotting data.

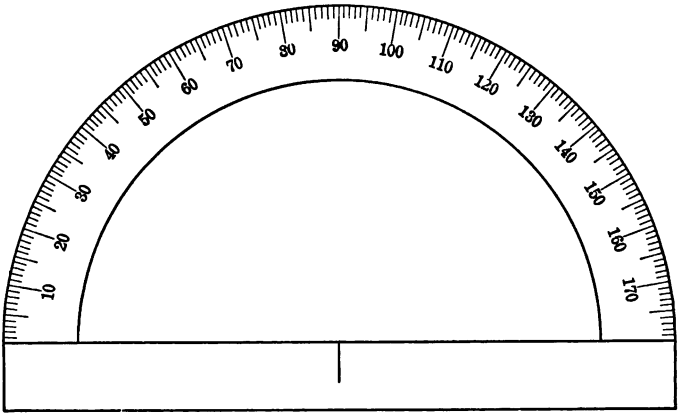


FIG. 4.—Protractor.

12. Twelve sheets of paper, form *M*7, used for numerical calculations and for tables.

13. Six sheets of polar coördinate paper, form *M*3.

14. Twenty-four sheets of heavy bond paper, letter size, 8½ by 11 inches; to be used for drawings.

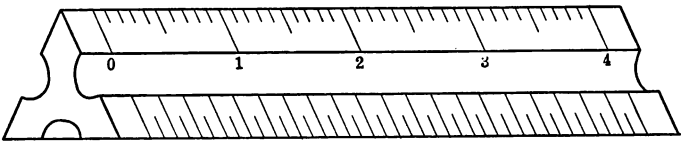


FIG. 5.—Triangular scale.

15. Two heavy paper note covers suitable for holding paper 8½ by 11 inches without folding. These covers are to be used for submitting daily work to the instructor.

16. One note cover large enough for filing 8½ by 11-inch sheets

without folding. This cover is to be used for filing the written work after having been examined and returned by the instructor.

17. A set of "Slichter's four-place logarithmic and trigonometric tables" used in connection with numerical computations.

There are other logarithmic and trigonometric tables published, but in order to do efficient classroom work without waste of time, there must be complete uniformity in the tables used by the members of the class.

If the student wishes larger tables for his *own* private use, the two following are among the best:

Albrecht's five-place, and
Bremiker's six-place.

While the above list of instruments and materials includes about the minimum in number and smallest in size for efficient work, more and better instruments may be purchased if the student desires.

2. General Directions. All work must be done *neatly*. In general, accuracy in graphic work can only be acquired by neat work. At first, perhaps, much time will be sacrificed, but after the habit of doing neat work has been established much time will be saved both in avoiding errors and in the ease with which errors may be located or the work checked. The student at the beginning must not be discouraged if a large number of his drawings and exercises are returned marked "*redraw*" or "*rewrite*," for neatness of work will always be insisted upon as one of the means of obtaining accurate work.

Unless special directions are given, all work submitted to the instructor is to be upon paper of letter size, 8½ by 11 inches, and enclosed without folding in a paper folder. These exercises when returned by the instructor are to be corrected in accordance with instructions, and kept in order in a note cover. Exercises marked "*rewrite*" or "*redraw* " are to be done over, and *both* the original and the corrected sheets filed. This file of exercises may be called for at any time by the instructor.

All *drawings* are to be done in pencil, unless the student has had training in the use of the ruling pen, in which case he may, if he desires, "ink in" his drawings with drawing ink.

All descriptive work upon a drawing is to be *lettered, not written.*[1] Each drawing or exercise handed in must be numbered in the upper right-hand corner. It must have a complete yet brief title, and contain a sufficient number of descriptive words in the body of the drawing so that anyone familiar with the problem will be able to read the drawing; *i.e.*, to understand the meaning of every line or figure in the drawing.

FIG. 6.—Set of letters and figures.

All exercises—*not drawings*—are to be done in ink. Use any good writing fluid. It is suggested that the student acquire the habit of working numerical problems, and the like, with pen and ink.

3. To draw a straight line through a point with the T-square or triangle as guide, place the triangle or T-square upon the paper, so that the point is just visible at one of its edges; press the triangle firmly against the paper with the fingers of the left hand and with the pencil in the right hand hold a flat side of the chisel-point against the straight edge and move the pencil from left to right. The pencil should be held with its upper end tipped slightly away

[1] A set of letters and figures is given in Fig. 6.

from the triangle in order to bring the point of the pencil in contact with the straight edge. (See Fig. 7.) As far as possible, the whole right hand should be given a straight-line motion in order that the pencil may be made to move parallel to itself.

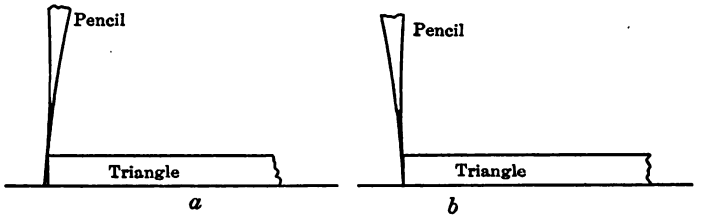


FIG. 7.—*a*, Incorrect way of holding a pencil; *b*, correct way of holding a pencil.

Care must be taken that the triangle is well pressed down upon the paper at the point where the line is being drawn. This is partly accomplished by the pressure of the fingers of the left hand, and partly by pressing down upon the triangle with the third and

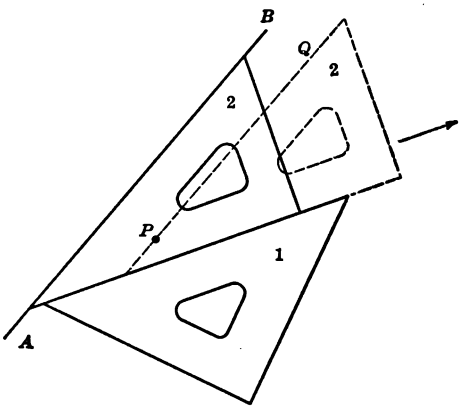


FIG. 8.

fourth fingers of the right hand. This pressing down with the fingers of the right hand not only helps to hold the triangle in place, but, at the same time, gives a rest to the hand.

4. Parallel Lines. *Horizontal parallel lines* may be drawn with the aid of the T-square.

Parallel lines making angles of 30°, 45°, 60° or 90° with the horizontal may be drawn with the aid of triangles placed upon the edge of the T-square.

Parallels other than those mentioned above may be drawn as follows: Suppose that through the point P, Fig. 8, we wish to draw a line parallel to the line AB. Place two triangles together, as shown in the figure, one edge of No. 2 coinciding with the given line. Hold triangle No. 1 fixed and slide No. 2 along it in the direction of the arrow until its edge which originally coincided with AB passes through the point P. Along this edge draw a line, PQ. PQ is parallel to AB, for, by geometry, if two straight lines are cut by a transversal, and if the interior exterior angles on the same side of the transversal are equal, the lines are parallel.

Exercises

1. Fasten a sheet of paper upon the drawing board with its longer dimension extending from left to right. About 2 inches from the lower edge of the sheet draw, with the aid of the T-square, a horizontal line 6 inches long. Divide this line into six equal parts. At the points of division draw vertical lines with the aid of the T-square and triangle. Beginning with the horizontal line mark off four equal spaces, each 1 inch long, upon the left-hand vertical line. From the points of division just located draw horizontal lines, thus forming a rectangle 4 by 6 inches, divided into twenty-four equal squares. Through the lower left-hand corner draw lines making angles of 30°, 45°, and 60° with the horizontal. Divide the upper left-hand square into 100 equal squares. A drawing (reduced) of this exercise is given in Fig. 9. Complete the exercise by giving, to two decimal places, the dimensions and quotients omitted in Fig. 9.

2. In a 45° triangle, prove by geometry that the hypotenuse is $\sqrt{2}$ times as long as a leg.

3. In a 60° triangle, prove that the hypotenuse is 2 times as long as the shorter leg. Prove that the longer leg is $\sqrt{3}$ times as long as the shorter leg. The hypotenuse is how many times as long as the longer leg?

4. Find, correct to three decimal places, $\sqrt{2}$; $\sqrt{3}$; $\sqrt{5}$.

5. If the hypotenuse of a 60° triangle is two units in length, what

is the length of the shorter leg? Of the longer leg? If the hypotenuse is four units in length, what is the length of each leg?

6. What is the length of the hypotenuse of a 45° triangle, if each

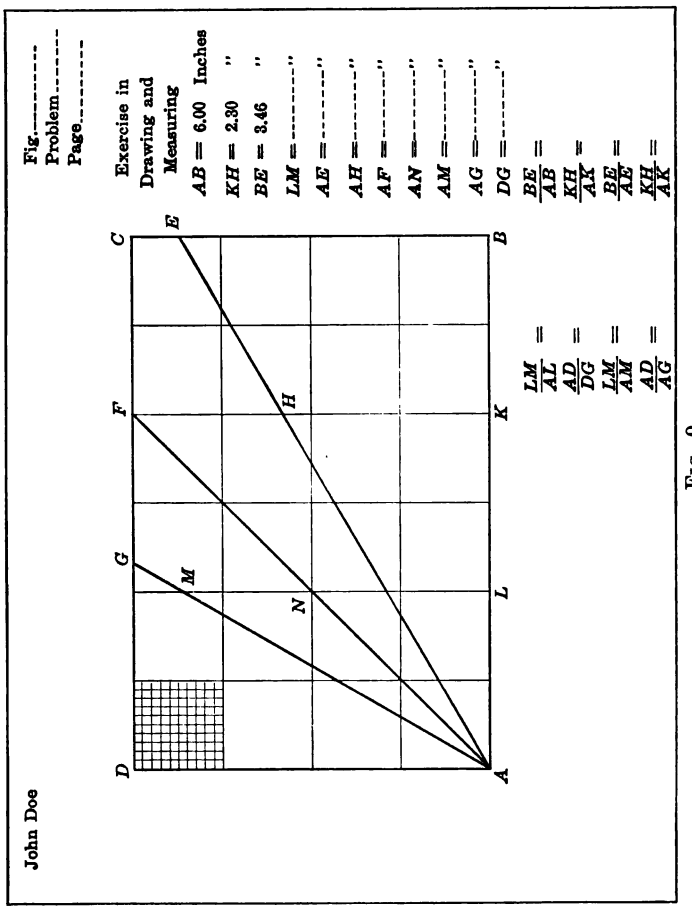


Fig. 9.

leg is a unit in length? Two units in length? Three units in length?

7. What is the length of each leg of a 45° triangle, if the hypotenuse is one unit in length? Two units in length? Three units in length?

8. Find the length in rods of a diagonal of a section of land, assuming that the section is a true square 1 mile on a side.

9. A, B, C, and D are the corners of a section of land. Through A, a line is drawn making an angle of 30° with the side AB, and intersecting BC at the point E. Find the length, in rods, of BE, and of AE. Find the area in acres of the triangular plot ABE.

10. Continuing with exercise 9, let F be the mid-point of AB, and G the mid-point of CD. Draw the quarter section line FG, cutting the line AE at H. Find the length in rods of AH, and of FH. Find the area in acres of the plot AFH.

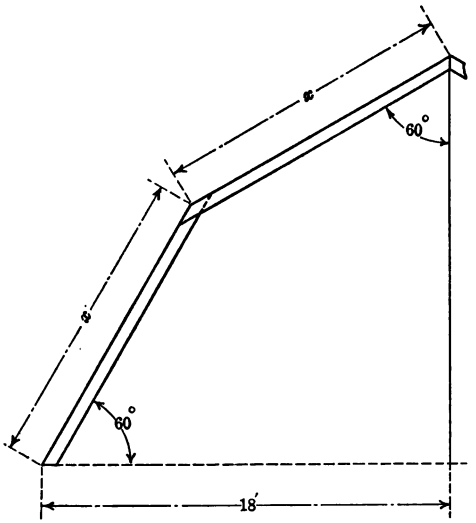


FIG. 10.

11. The longer leg of a plot of land in the form of a 60° triangle is 80 rods. Find the area of the plot in acres.

12. A plot of land in the form of a 60° triangle contains 72 acres. Find the length in rods of each side of the triangle.

HINT: Let x represent the number of rods in the length of the shorter leg.

13. The shorter side of a rectangle is 100 feet, the diagonal is 200 feet. Find the length of the longer side.

14. The diagonal of a square is 100 feet. Find the length of a side.

15. A barn 36 feet wide has a gambrel roof. The lower rafters make an angle of 60° with the horizontal. The upper rafters make an angle of 60° with the vertical. The lower and upper rafters are equal in length. Find the length of the upper edge of the rafters. See Fig. 10.

HINT: Let x be the number of feet in the desired length.

16. If the rafters referred to in exercise 15 are 2 × 6's, and if they run short 3/16 inch, find the dimensions of the block sawed from each end. (See Fig. 11.) Find y and z.

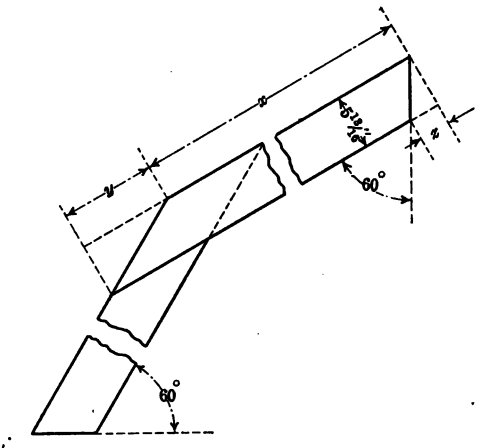


FIG. 11.

17. If the building referred to in exercise 15 is 84 feet long, find the number of cubic yards space under the roof above the plates.

18. A circular concrete silo, 12 feet outside diameter, is 20 feet tall. The wall is 8 inches thick. Find the number of cubic yards of concrete in the wall, no allowance being made for openings. After the problem has been solved by the student, the instructor will show how algebraic operations may be used in eliminating considerable numerical work.

19. A man walks 3/4 mile up an incline making an angle of 30° with the horizontal. How many feet does he move in a horizontal direction?

20. A ball rolls down a smooth inclined plane making an angle of 45° with the horizontal. If the distance s measured in feet along the incline is given by the formula

$$s = 11.4t^2$$

where t is time in seconds, find the vertical distance through which the ball moves in one second; in two seconds; in three seconds.

5. Review of Elementary Algebra. The course for which this book is intended presupposes a knowledge of elementary algebra, and of plane and solid geometry, as usually covered in two years of High School work. Many students, however, having had their algebra and geometry some years since, may feel a need of review in these subjects. When this is the case, the student should take a hasty review touching only the essentials, and then study those subjects or theorems which he wishes to apply, at the time he wishes to apply them. For this purpose, a very brief review of algebra, consisting mostly of examples and illustrations, is here given.

6. Addition.

To	$3x + 4y - 6z$
Add	$2x - 5y + 2z$
Sum	$5x - y - 4z$

Exercises

Find the sum of the following:

1. $3a - 6b + 7c$ and $5a - 2c - 7b$.

2. $3x^2 + 6xy + y^2$ and $x^2 - 5xy + y^2$.

3. $2xy - 3x^2y + 3xy^2 + y^2$ and $2x^2y - 3y^2 + 3xy - 7$.

4. $a + b + c - d$ and $2a - 7c - 2d + 6b - 7$.

5. $a - 2b + 6$, $3a + 7c - 6 + 2b$, and $5a + 6 - 7c$.

6. $x^2 + 2xy + y^2$, $x^2 - 2xy + y^2$, and $x^2 - 2y^2$.

7. $3x^3 + 6x^2y + 6xy^2 + y^3$, $5x^3 - 6y^3 + 6x^2y$, and $3x^3 - y^3 - 7xy^2 - 9x^2y$.

8. $a + b + c + 3d$, $2a - c + 7b - 5d + 6$, and $c - 7 + 2d - a$.

9. $a^3 - b^3$, $a^2b + ab^2$, and $-4a^2b + 2ab^2$.

10. $25a^3 - 27b^3$, $5a^2b - 6ab^2$, and $26b^3 - 24a^3 + ab^2$.

7. Subtraction.

From	$3a - 6b + 7c - 6$
Subtract	$2a + 5b - 2c - 6$
Difference	$a - 11b + 9c$

Exercises

1. From $3a^2 - 2b^2 - 3ab$ subtract $a^2 - 3ab + b^2$.
2. From $x^2 - 2xy + y^2$ subtract $x^2 + 2xy + y^2$.
3. From $x + 2x^2 + 3x^3 - 2$ subtract $2x + x^2 + 7x^3$.
4. From $x^3 + 3x^2y + 3xy^2 + y^3$ substract $6x^2y + 2y^3$.
5. From $3x^2 - 2y + y^2$ subtract $2x^2 - 2y + 6 + 7x^2$.

8. Multiplication. The product of two numbers having like signs, either positive or negative, is positive. The product of two numbers having unlike signs is negative. Thus,

$$(a)(b) = ab$$
$$(-a)(-b) = ab$$
$$(a)(-b) = -ab$$
$$(-a)(b) = -ab$$

$$\begin{array}{ll} \text{Multiply} & 3x^2 - 2xy + y^2 \\ \text{By} & 2x^2 - xy + y^2 \\ \hline & 6x^4 - 4x^3y + 2x^2y^2 \\ & \quad - 3x^3y + 2x^2y^2 - xy^3 \\ & \qquad\qquad 3x^2y^2 - 2xy^3 + y^4 \\ \hline \text{Product} & 6x^4 - 7x^3y + 7x^2y^2 - 3xy^3 + y^4 \end{array}$$

Exercises

1. Multiply $x^3 + xy + y^3$ by $x - y$.
2. Multiply $x^3 - xy + y^3$ by $x + y$.
3. Multiply $x^2 + xy + y^2$ by $x^2 - xy + y^2$.
4. Multiply $x - 7y$ by $x - 2y$.
5. Multiply $x + y$ by $x - y$.
6. Multiply $3x + 7y$ by $3x + 8y$.
7. Multiply $a + b + c + d$ by $a + b - c - d$.
8. Multiply $2x - 3y + 6$ by $2x + 7y + 2$.
9. Multiply $3xy - 2x^2 - 2x$ by $2xy - 2x - 3y - 6$.
10. Multiply $9 - 3x + x^2$ by $3 + x$.

A few simple multiplications may be performed mentally. We see at once that $(a + b)(a - b) = a^2 - b^2$, where a and b are any two numbers. From this we have

$$(3x - 2y)(3x + 2y) = 9x^2 - 4y^2$$

Exercises

Multiply the following mentally:

1. $(3x - y)(3x + y)$.
2. $(2x - y)(2x + y)$.
3. $(a - 9)(a + 9)$.
4. $(x^2y - 3a)(x^2y + 3a)$.
5. $(a + 3b)(a - 3b)$.
6. $(29)(31)$, or $(30 - 1)(30 + 1) = 900 - 1 = 899$.
7. $(51)(49)$.
8. $(52)(48)$, or $(50 + 2)(50 - 2)$.
9. $(103)(97)$.
10. $(25)(35)$.

A few powers of binomials are:

$$(a + b)^2 = a^2 + 2ab + b^2.$$
$$(a - b)^2 = a^2 - 2ab + b^2.$$
$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$
$$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3.$$
$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$
$$(a - b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4.$$

Thus, $(3 - a)^3 = 27 - 27a + 9a^2 - a^3$,

and $(x + y^2)^4 = x^4 + 4x^3y^2 + 6x^2y^4 + 4xy^6 + y^8$.

Exercises

Expand the following mentally:

1. $(2a - x)^2$.
2. $(x + 3y)^2$.
3. $(2x - 1)^3$.
4. $(x - 3)^4$.
5. $(1 - x)^3$.
6. $(2 + y)^4$.
7. $(52)^2$, or $(50 + 2)^2$, or $2500 + 200 + 4 = 2704$.
8. $(31)^2$, or $(30 + 1)^2$.
9. $(29)^2$, or $(30 - 1)^2$.
10. $(98)^2$.

The square of a polynomial is illustrated by the following:

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.$$
$$(a + b + c + d)^2 = a^2 + b^2 + c^2 + d^2 + 2ab + 2ac + 2ad + 2bc + 2bd + 2cd.$$
$$(3 - x + y^2)^2 = 9 + x^2 + y^4 - 6x + 6y^2 - 2xy^2.$$

Exercises

Expand the following mentally:

1. $(a + b + 2)^2$.
2. $(a + b - 2)^2$.
3. $(a - b - c)^2$.
4. $(2a - x + 3)^2$.
5. $(x^2 - 2y^2 + 4)^2$.

The product of two binomials having a common term may be found by inspection. Illustrations:

$$(a + b)(a + c) = a^2 + (b + c)a + bc.$$
$$(x + 7)(x + 2) = x^2 + 9x + 14.$$
$$(x - 5)(x + 3) = x^2 - 2x - 15.$$
$$(x^2 - 2y)(x^2 - 3y) = x^4 - 5x^2y + 6y^2.$$

Exercises

Find mentally the value of each of the following:

1. $(x + 2)(x + 3)$.
2. $(x - 2)(x + 3)$.
3. $(x - 2)(x - 3)$.
4. $(x + 2)(x - 3)$.
5. $(x^2 + 5y)(x^2 - 5y)$.
6. $(3x + 2y)(3x - 7y)$.
7. $(x^2 - 3)(x^2 - 4)$.
8. $(3xy - z)(3xy + 7z)$.
9. $(x^2y^2 - 3)(x^2y^2 - 10)$.
10. $(x - 2y)(2x - 2y)$.

9. Division. The quotient of two numbers having like signs is positive. The quotient of two numbers having unlike signs is negative.

Thus,
$$(a) \div (b) = \frac{a}{b}$$
$$(-a) \div (-b) = \frac{a}{b}$$
$$(a) \div (-b) = -\frac{a}{b}$$
$$(-a) \div (b) = -\frac{a}{b}$$

Divide $x^3 - 6x^2 - 19x + 84$ by $x - 7$

$$\begin{array}{l} x - 7 \overline{)x^3 - 6x^2 - 19x + 84} \quad x^2 + x - 12 \quad \text{Quotient} \\ \qquad x^3 - 7x^2 \\ \qquad \overline{\qquad x^2 - 19x + 84} \\ \qquad x^2 - 7x \\ \qquad \overline{\qquad -12x + 84} \\ \qquad -12x + 84 \\ \qquad \overline{\qquad\qquad} \end{array}$$

Exercises

1. Divide $15x^2 - 11x - 14$ by $3x + 2$.
2. Divide $12a^2 - 28a + 15$ by $6a - 5$.
3. Divide $8m^5 - 14m^2 - 18m + 21$ by $4m^3 + 6m - 7$.
4. Divide $3x^5 - 7x^3 - 2x - 36$ by $x - 2$.
5. Divide $x^6 + 2x^5 - 3x^4 + 7x^3 - 2x + 69$ by $x - 3$.
6. Divide $x^4 + x^2y^2 + y^4$ by $x^2 + xy + y^2$.
7. Divide $8x^4 - 22x^3y + 43x^2y^2 - 38xy^3 + 24y^4$ by $2x^2 - 3xy + 4y^2$.
8. Divide $x^3 - y^3$ by $x - y$.
9. Divide $x^4 - y^4$ by $x + y$.
10. Divide $x^4 - y^4$ by $x - y$.

10. Symbols of Aggregation. If a sign of aggregation is preceded by the negative sign, change all signs within when the sign of aggregation is removed. If the sign of aggregation is preceded by the positive sign, all signs within remain the same when the sign of aggregation is removed.

$$\begin{aligned} 5x^2 - [3y^2 + \{2x^2 - (y^2 + 3x^2) + 5y^2\} - x^2] \\ = 5x^2 - [3y^2 + \{2x^2 - y^2 - 3x^2 + 5y^2\} - x^2] \\ = 5x^2 - [3y^2 + \{4y^2 - x^2\} - x^2] \\ = 5x^2 - [3y^2 + 4y^2 - x^2 - x^2] \\ = 5x^2 - [7y^2 - 2x^2] \\ = 5x^2 - 7y^2 + 2x^2 \\ = 7x^2 - 7y^2 \end{aligned}$$

Exercises

Simplify the following by removing the signs of aggregation:

1. $ab - 4b^2 - (2a^2 - b^2) - \{-5a^2 + 2ab - 3b^2\}$.
2. $x - \{y + z - [x - (-x - y) + z]\} + [z - (2x - y)]$.
3. $a - \{-a - [-a - (-a - 1)]\}$.
4. $3yz - [2yz + (9z - 2yz)]$.
5. $-\{-1 - [-1 - (-1)]\}$.

11. Factoring. Since $(a \pm b)^2 = a^2 \pm 2ab + b^2$, any expression of the form of the right-hand side can be factored by inspection. Thus,

$$x^2 - 6xy + 9y^2 = (x - 3y)^2$$

and

$$4 + 4(a + b) + (a + b)^2 = (2 + a + b)^2$$

Exercises

Factor the following by inspection:

1. $9x^2 - 30xy + 25y^2$.
2. $4 + 16t + 16t^2$.
3. $x^4y^4 + 10x^2y^2z^2 + 25z^4$.
4. $9 + 6(x^2 + y^2) + (x^2 + y^2)^2$.
5. $a^4 + 4a^2b^2 + 4b^4$.

Since $(a + b)(a - b) = a^2 - b^2$, any expression of the form of the right-hand side can be factored by inspection. Thus,

$$4a^2 - 9b^2 = (2a + 3b)(2a - 3b)$$

Exercises

Factor the following by inspection:

1. $x^2y^2 - z^2$.
2. $(a + b)^2 - c^2$.
3. $c^2 - (a + b)^2$.
4. $25 - 3x^2$.
5. $81 - 625x^4$.

Since $(a + b)(a + c) = a^2 + (b + c)a + bc$, any expression of the form of the right-hand side can be factored by inspection. Thus,

$$x^2 - 5x - 14 = (x - 7)(x + 2)$$

Exercises

Factor the following by inspection:

1. $x^2 + 7x + 10$.
2. $a^2 + 4ay - 21y^2$.
3. $4x^2 - 18xy + 18y^2$.
4. $9x^2 - 18x - 27$.
5. $25 + 30a - 27a^2$.

Since $(a + b)(a^2 - ab + b^2) = a^3 + b^3$, any expression of the form of the right-hand side can be factored by inspection. Thus,

$$27 + 125x^3 = (3 + 5x)(9 - 15x + 25x^2)$$

Exercises

Factor the following by inspection:

1. $x^3y^3 + 1$.
2. $x^6 + y^6$.
3. $8 + 27x^3$.
4. $125 + x^3y^3$.
5. $x^3 + 8y^6$.

Since $(a - b)(a^2 + ab + b^2) = a^3 - b^3$, any expression of the form of the right-hand side can be factored by inspection. Thus,

$$27 - 125x^3 = (3 - 5x)(9 + 15x + 25x^2)$$

Exercises

Factor the following by inspection:

1. $x^2y^3 - 1$.
2. $x^6 - y^6$, or $(x^3 + y^3)(x^3 - y^3)$.
3. $8 - 9x^3$.
4. $125 - x^2y^2$.
5. $x^3 - 8y^6$.

The following may be factored by grouping the terms. Thus,

$$\begin{aligned} & a^3m + a^3n - m - n \\ &= a^3(m + n) - (m + n) \\ &= (a^3 - 1)(m + n) \\ &= (a - 1)(a^2 + a + 1)(m + n) \end{aligned}$$

Exercises

Factor the following:

1. $ax - ay + bx - by$.
2. $x^3 + 3x^2 + 3x + 1$.
3. $ax^2 - 2axy + ay^2 + bx^2 - 2bxy + by^2$.
4. $x^5 - xy^4 - x^4y + y^5$.
5. $x^4 - x^3y - xy^3 + y^4$.

12. Fractions. Multiplying or dividing both numerator and denominator of a fraction by the same number, excepting zero, does not change the value of the fraction. To reduce a fraction to its lowest terms, factor both numerator and denominator and then divide each by the common factors if there are any. Thus,

$$\begin{aligned} & \frac{ax^4 + ax^2y^2 + ay^4}{a^2x^3 - a^2y^3} \\ &= \frac{a(x^2 + xy + y^2)(x^2 - xy + y^2)}{a(x - y)(x^2 + xy + y^2)} \\ &= \frac{x^2 - xy + y^2}{(x - y)} \end{aligned}$$

Exercises

Reduce the following to lower terms:

1. $\dfrac{ax^2 - ay^2}{x^3 - y^3}$.
2. $\dfrac{x^2 + x - 6}{x^2 - 4}$.
3. $\dfrac{x^4 - y^4}{x^6 - y^6}$.
4. $\dfrac{ax + ay - x - y}{x^3 + y^3}$.
5. $\dfrac{27 - x^3}{12 - 7x + x^2}$.

To multiply fractions together, multiply the numerators together for the numerator of the product, and multiply the denominators together for the denominator of the product. To simplify the result, divide both numerator and denominator by common factors if there are any. Thus,

$$\frac{a^3 - b^3}{ab} \times \frac{b}{a^2 + ab + b^2} \times \frac{a^2b - ab^2}{(a - b)^2}$$
$$= \frac{(a^3 - b^3)(b)(a^2b - ab^2)}{ab(a^2 + ab + b^2)(a - b)^2}$$
$$= \frac{(a - b)(a^2 + ab + b^2)(ab^2)(a - b)}{ab(a^2 + ab + b^2)(a - b)^2}$$
$$= b$$

In working this problem, the second expression should not be written down, and possibly the third could be omitted.

Exercises

Simplify the following:

1. $\dfrac{a^4 - y^4}{a^2 - 2ay + y^2}$.

2. $\dfrac{a^2(x + a)}{x^2 + 2ax + a}$.

3. $\dfrac{a^2 + 2ab + b^2 - 4c^2}{4a^2 - (b + 2c)^2} \times \dfrac{(a - b)^2 - 4c^2}{(a + b)^2 - 4c^2} \times \dfrac{(2a - b)^2 - 4c^2}{a^2 - (b - 2c)^2}$.

4. $\dfrac{x^3 - y^3}{x^3 + y^3} \times \dfrac{x^2 - xy + y^2}{x^2 + xy + y^2} \times \dfrac{x + y}{x - y}$.

5. $\dfrac{ax + ay}{b} \times \dfrac{bx - 3b}{x^3 + y^3} \times \dfrac{x^2 - xy + y^2}{x^2 - 5x + 6}$.

To add fractions reduce each to the same denominator by multiplying numerator and denominator of each by a suitable number, then add the numerators and place the sum over the common denominator. Thus,

$$\frac{x - y}{x^2 - xy + y^2} + \frac{1}{x + y} + \frac{xy}{x^3 + y^3}$$
$$= \frac{x^2 - y^2}{x^3 + y^3} + \frac{x^2 - xy + y^2}{x^3 + y^3} + \frac{xy}{x^3 + y^3}$$
$$= \frac{2x^2}{x^3 + y^3}$$

Exercises

Simplify the following:

1. $\dfrac{7}{x+2}+\dfrac{5}{x-3}+\dfrac{1}{x^2-x-6}$.
2. $\dfrac{1}{y+1}-\dfrac{y-2}{y^2-y+1}-\dfrac{3}{y^3+1}$.
3. $\dfrac{1}{y^2-1}+\dfrac{1}{(y+1)^2}+\dfrac{y(y^2+3)}{(y^2-1)^2}-\dfrac{y}{(y-1)^3}$.
4. $\dfrac{x-5}{x-1}-\dfrac{x-1}{x-5}+\dfrac{5}{x^2-6x+5}$.
5. $3-\dfrac{6(x+y)}{x-y}+\dfrac{(x+y)^2}{(x-y)^2}$.

13. Exponents,

Definitions: $\mathbf{a}^n = \mathbf{a}\,\mathbf{a}\,\mathbf{a}\,\ldots$ to n factors, if n is a positive integer

$$\mathbf{a}^{p/q} = \sqrt[q]{\mathbf{a}^p}, \text{ if } p \text{ and } q \text{ are positive integers}$$

$$\mathbf{a}^0 = \mathbf{1}$$

$$\mathbf{a}^{-s} = \frac{\mathbf{1}}{\mathbf{a}^s}, \text{ if } s \text{ is positive}$$

Laws of indices: $\mathbf{a}^m\mathbf{a}^n = \mathbf{a}^{m+n}$

$$\frac{\mathbf{a}^m}{\mathbf{a}^n} = \mathbf{a}^{m-n}$$

$$(\mathbf{a}^m)^n = \mathbf{a}^{mn}$$

$$\sqrt[q]{\mathbf{a}^p} = \left[\sqrt[q]{\mathbf{a}}\right]^p$$

Exercises

1. Express the following using positive exponents only:

(*a*) $x^{-2}ab^{-3}$.

(*b*) $a^{-\frac{1}{3}}b^{\frac{2}{3}}$.

(*c*) $(x^{-\frac{1}{2}}y^{-\frac{2}{3}})^6$.

(*d*) $\dfrac{a^{-1}b^{-2}x^{-3}}{c^2a^3x^5}$.

(*e*) $\dfrac{a^{-1}bc^{-2}}{ab^{-2}c^{-1}}$.

2. Perform the indicated operations:

(*a*) $25^{\frac{1}{2}}$. (*b*) $8^{\frac{2}{3}}$. (*c*) $(9/16)^{-\frac{1}{2}}$. (*d*) 27^0.

3. Simplify:

(*a*) $\left[\dfrac{a^{-\frac{2}{3}}b^2}{b^{\frac{1}{6}}}\right]^{\frac{1}{3}}$.

(*b*) $\left[\dfrac{x^0y^{-\frac{1}{2}}}{x^2y^{-\frac{7}{2}}}\right]^{\frac{2}{3}}$.

4. Multiply $8a^{3\frac{1}{2}} - 4ab^{1\frac{1}{2}} + 2a^{1\frac{1}{2}}b - b^{3\frac{1}{2}}$ by $2a^{1\frac{1}{2}} - 3b^{1\frac{1}{2}}$.

5. Divide $a^{7\frac{1}{3}} - 3a^2 + a^{5\frac{1}{3}} - 4 + 12a^{1\frac{1}{3}} - 4a^{3\frac{1}{3}}$ by $a^{5\frac{1}{3}} - 4$.

14. Radicals.

Simplify $\sqrt[3]{24} + 5\sqrt[3]{81} - 6\sqrt[3]{192} - 10\sqrt[3]{1029}$

$$= \sqrt[3]{2^3 3} + 5\sqrt[3]{3^3 3} - 6\sqrt[3]{4^3 3} - 10\sqrt[3]{7^3 3}$$
$$= 2\sqrt[3]{3} + 15\sqrt[3]{3} - 24\sqrt[3]{3} - 70\sqrt[3]{3}$$
$$= -77\sqrt[3]{3}$$

Rationalize the denominator of

$$\frac{\sqrt{2}+\sqrt{3}}{\sqrt{2}-\sqrt{3}} = \frac{(\sqrt{2}+\sqrt{3})(\sqrt{2}+\sqrt{3})}{(\sqrt{2}-\sqrt{3})(\sqrt{2}+\sqrt{3})}$$
$$= \frac{(\sqrt{2}+\sqrt{3})^2}{2-3}$$
$$= \frac{2+2\sqrt{6}+3}{-1}$$
$$= -(5+2\sqrt{6})$$

Exercises

1. Simplify $\sqrt{12} - 2\sqrt{27} + 9\sqrt{48} - \sqrt{108} - 6\sqrt{75}$.
2. Simplify $\sqrt{(a-b)^2x} - \sqrt{(a+b)^2x} - \sqrt{a^2x} + \sqrt{b^2x}$.
3. Rationalize the denominator of $\frac{2}{\sqrt{3}}$.
4. Rationalize the denominator of $\frac{1}{\sqrt{3}-\sqrt{2}}$.
5. Rationalize the denominator of $\frac{2-\sqrt{5}}{\sqrt{3}-\sqrt{5}}$.

15. Quadratic Equations. A quadratic equation may be solved by completing the square, or by using the formula, or somtimes by factoring.

Solve for x: $x^2 - 6x + 8 = 0$.

By completing the square we have

$$x^2 - 6x + 9 = 1,$$
or $$x - 3 = \pm 1,$$
or $$x = 3 \pm 1,$$
or $$x = 4 \text{ or } 2.$$

By factoring the left-hand side we have

$$(x - 4)(x - 2) = 0.$$

This equation is equivalent to the two equations.

$$x - 4 = 0$$

and

$$x - 2 = 0$$

from which we have

$$x = 4$$

and $$x = 2$$

If $ax^2 + bx + c = 0$ is the general equation of the second degree, its solutions are

$$\mathbf{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

In the above equation, $a = 1$, $b = -6$, and $c = 8$. Substituting these values in the formula we have

$$x = \frac{6 \pm \sqrt{36 - 4 \cdot 8}}{2}$$

$$= \frac{6 \pm 2\sqrt{9 - 8}}{2.}$$

$$= 3 \pm 1,$$

or $$x = 4 \text{ or } 2$$

Exercises

Solve the following equations for x:

1. $x^2 - 9x + 20 = 0$.
2. $4x^2 + 7x - 2 = 0$.
3. $x^2 - 7x + 6 = 0$.
4. $8x^2 - 4x + 3 = 0$.
5. $9x^2 + 7x - 20 = 0$.

CHAPTER I

GRAPHIC REPRESENTATION

The Straight Line. The Parabola. The Equilateral Hyperbola.

16. The Scale. In solving problems graphically the fundamental instrument used is the scale. Unless otherwise stated, the term scale, as used in this book, means what is commonly understood by a scale, an instrument for measuring short linear distances. The term scale, in a broader sense, may be defined as any curve, including a straight line, the points along which are supposed to correspond in order to the numbers of a sequence. We are all familiar with various scales, such as the foot-rule, divided into inches; the meter, divided into centimeters and millimeters; the thermometer scale, divided to correspond to degrees and tenths of degrees of temperature; the scale on a burette, divided to correspond to cubic centimeters and tenths of cubic centimeters; and scores of other scales used for measuring different quantities. If a scale is marked so that the subdivisions are all of equal length, as the foot-rule, we say the scale is divided uniformly, or that we have a **uniform scale;** otherwise, the scale is a **non-uniform scale.** For the solution of problems graphically, the student should have for measuring lengths a scale divided into centimeters and millimeters or into inches and tenths of inches. Illustrations of non-uniform scales will be given later.

17. Drawing to Scale. A large number of graphic problems consist in nothing more than drawing to scale, *i.e.*, making a picture of some object in which a definite length, as the inch, on the drawing represents a definite length, as the foot, on the given object. If the dimensions on such a drawing were to be given in feet and inches, or if dimensions in feet and inches were to be scaled from the drawing, a scale divided into tenths of inches would apparently be very ill-adapted. For this kind of work, scales may be procured on which the unit of length is sub-

divided into twelve equal parts. If each unit throughout the entire length of the scale is subdivided, the scale is called a **full divided** or **chain scale.** If only one unit, usually the left-hand unit, is subdivided, it is called an **open divided scale.** These scales are made with the unit $\frac{1}{8}$, $\frac{1}{4}$, $\frac{3}{8}$, $\frac{1}{2}$, $\frac{3}{4}$, 1, $\frac{3}{2}$, 2, 3, etc., inches long, used, respectively, when $\frac{1}{8}$, $\frac{1}{4}$, $\frac{3}{8}$, $\frac{1}{2}$, $\frac{3}{4}$, 1, $\frac{3}{2}$, 2, 3, etc., inches on the drawing are to represent 1 foot on the object.

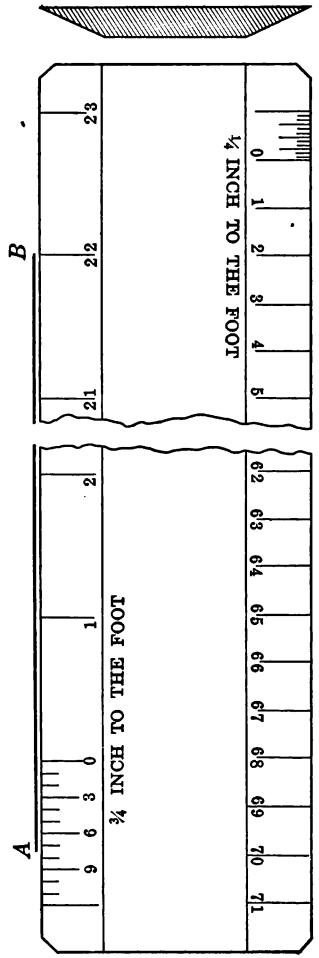


FIG. 12.—Open divided scale.

Fig. 12 represents an 18-inch, flat, open divided scale. The unit on one edge is $\frac{3}{4}$ inch, and on the other edge it is $\frac{1}{4}$ inch. This scale may be used in drawing the plans of a barn. The small scale is used in drawing the plan and elevations $\frac{1}{4}$ inch to the foot, while the larger scale is used in drawing, on a larger scale, the details of the plank frame truss bents. The method of measuring lines is illustrated in the figure. The line AB is found to be 22 feet, $7\frac{1}{2}$ inches long.

18. Positive and Negative Directions. If upon a scale some point is chosen as zero, or origin, distances measured in one direction, or the numbers represented by the points on one side of it, are called **positive,** and are designated with the positive, or plus, sign +. Distances measured in the other

direction, or numbers represented by points on the other side of the origin, are called **negative,** and are designated by the negative, or minus, sign −.

19. Rectangular coördinate paper, or squared paper, is paper ruled in both directions, horizontally and vertically. These rulings are generally spaced uniformly, dividing the page into a number of equal squares or rectangles. The size of these squares or rectangles is entirely arbitrary. The length of a side may be an inch, a tenth of an inch, a centimeter, a millimeter, etc.,

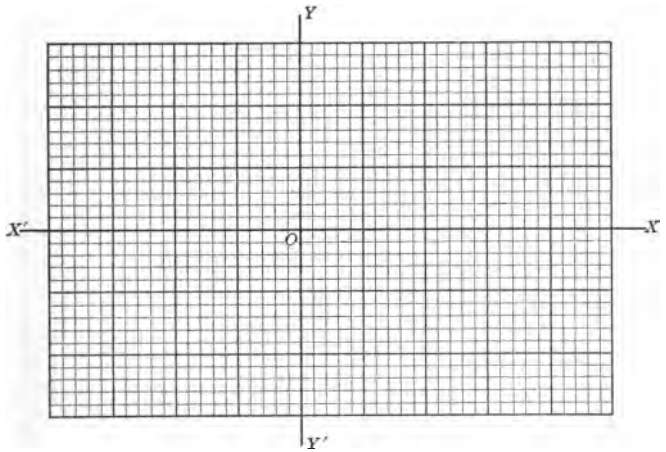


FIG. 13.—Coördinate, or squared paper.

depending, to some extent, upon the use for which the paper is designed. For convenience, every fifth, tenth, fourth, eighth, or twelfth line is made somewhat heavier than the others. A sheet of squared paper is represented in Fig. 13. Upon this sheet two lines, mutually at right angles, are chosen as **axes.** The horizontal, $X'OX$, is called the axis of **abscissas;** the vertical, $Y'OY$, the axis of **ordinates.** The point of intersection, O, of these axes is called the **origin.**

A use of squared paper is illustrated by the following: In Table I are given the maximum and minimum air temperatures at Madison, Wisconsin, for each day of the month of October, 1910.

TABLE I.—MAXIMUM AND MINIMUM TEMPERATURES AT MADISON WISCONSIN, FOR OCTOBER, 1910

Date	Max. temp., ° F.	Min. temp., ° F.	Date	Max. temp., ° F.	Min. temp. ° F.
1	66	55	17	81	53
2	71	48	18	81	57
3	75	58	19	69	45
4	68	55	20	45	40
5	62	53	21	49	41
6	58	45	22	52	34
7	66	43	23	60	37
8	68	47	24	60	49
9	60	44	25	52	44
10	67	42	26	60	40
11	75	49	27	42	32
12	61	46	28	35	30
13	69	45	29	38	26
14	73	52	30	60	31
15	76	50	31	63	39
16	80	56			

These data are said to be represented graphically in Fig. 14, or the data are said to be plotted upon squared paper. To do this: first, points equidistant along the axis of abscissas are chosen to represent the days of the month. These points are numbered consecutively from 1 to 31.[1] Distances from this horizontal line represent temperatures. If the temperature is above zero, or (+), it is represented by a distance above the horizontal axis; if below zero, or (−), by a distance below. The distance above or below is proportional to the magnitude of the temperature reading. Since distances from the horizontal axis represent temperature readings, points along the axis of ordinates are numbered to correspond to degrees of temperature. In the figure, the vertical distance between two adjacent horizontal lines represents 2°, but only every fifth line is actually numbered, since numbering each line would not aid in any way in reading the drawing. This vertical axis is in reality a uniform scale, for equal

[1] Only the even numbers are placed in the drawing; otherwise the numbers would be so near together as to be confusing. Omitting the numbers from every other point does not make the drawing harder to read.

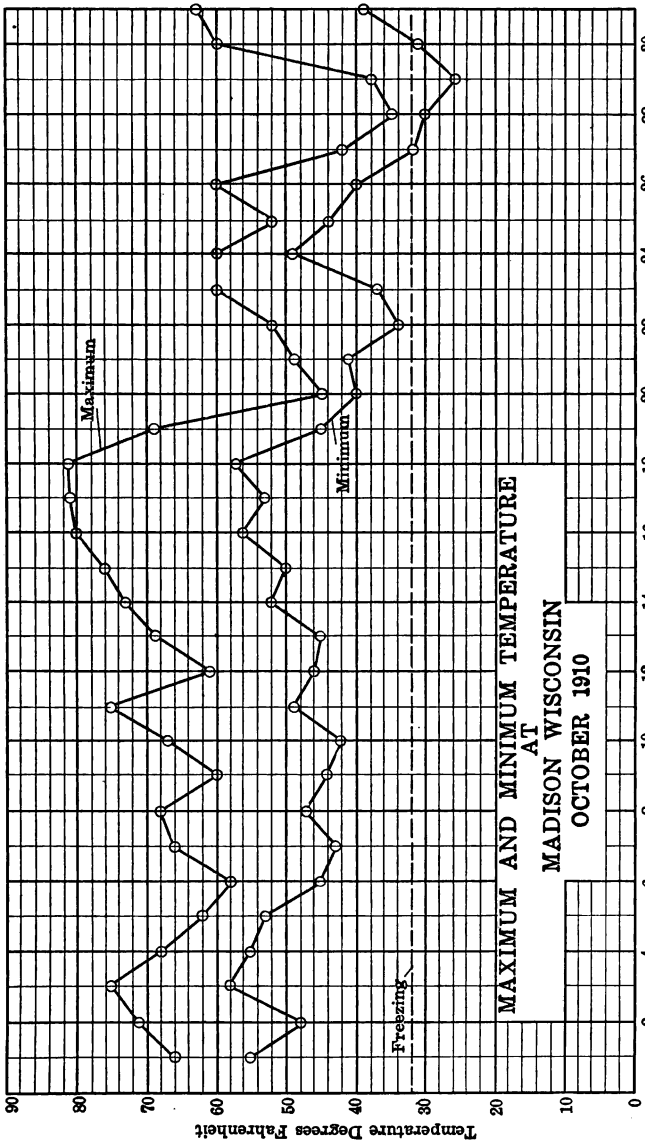

MAXIMUM AND MINIMUM TEMPERATURE
AT
MADISON WISCONSIN
OCTOBER 1910
Maximum
Minimum
Freezing
Temperature Degrees Fahrenheit
Days of Month October 1910
0
10
20
30
40
50
60
70
80
90
2
4
6
8
10
12
14
16
18
20
22
24
26
28
30

FIG. 14.

portions of it represent equal changes in temperature. Since the scale is uniform, it is evident that it is immaterial whether we think of the point or the vertical distance from the origin to that point as representing degrees of temperature. Further, we may look upon the scale as continuous, for *every* point upon it corresponds to some temperature.

The horizontal axis, as we are here using it, is not a continuous scale for measuring time, for each numbered point along it, or the distance of this point from the origin, represents a whole interval of time from midnight to midnight; and the intermediate points have no meaning whatever.

Now that we have chosen points to represent the days of the month, a unit of length to represent a degree of temperature, and have marked the axes, we are ready to transfer the readings from the table to the squared paper. On the first of October, the maximum temperature was + 66°. Graphically, this is represented by making a dot or small circle thirty-three divisions above the horizontal axis and upon the vertical line passing through that point of the horizontal axis representing the first day of October. This reading is now said to be **plotted.** The maximum temperature on the second of October was 71°, and is represented by placing a point or circle thirty-five and one-half spaces above the horizontal axis and upon a vertical line passing through that point upon the horizontal axis marked 2. In a similar way, points representing the maximum and minimum temperatures for each day are plotted. To aid the eye in following the temperatures from day to day the consecutive points are connected by short straight lines. These broken lines will also aid one to see at a glance how the maximum and minimum temperatures rose and fell from day to day throughout the month; to see when the temperature was high; when low; when the freezing point was reached; etc. In short, Fig. 14 is a picture of all that is given in Table I.

20. Smooth Curves Drawn Through Plotted Points. In Fig. 15 are plotted the data given in Table II, the hourly air temperatures for May 14 and October 10, 1910, at Madison, Wisconsin. It will be noticed that in this drawing the points are connected, not by straight lines as in the previous illustration, but by means

of a smooth curve drawn through the points. The reason for drawing a smooth curve is that the readings were taken so near together that the points along the smooth curve are probably very near the points that would have been obtained had additional intermediate temperature readings been taken. Thus the curves in Fig. 15 may be used for scaling off temperatures for any time of the day.

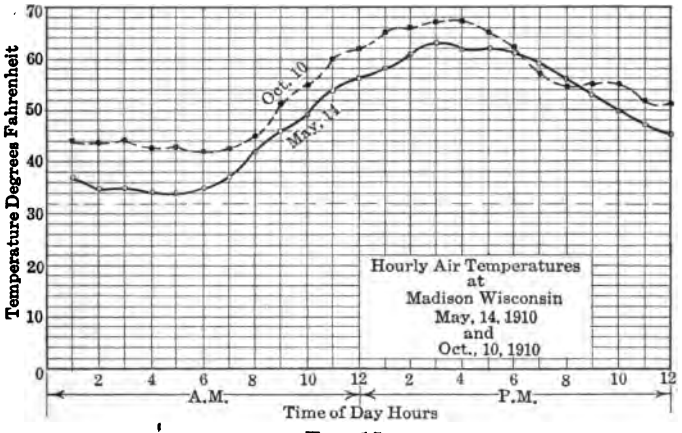


FIG. 15.

TABLE II.—HOURLY AIR TEMPERATURES AT MADISON, WISCONSIN, MAY 14, 1910; OCTOBER, 10, 1910

Time	May 14, 1910, temp., ° F.	Oct. 10, 1910, temp., ° F.	Time	May 14, 1910, temp., ° F.	Oct. 10, 1910, Temp., ° F.
1 a. m.	37	44	1 p. m.	58	65
2 a. m.	35	44	2 p. m.	61	66
3 a. m.	35	44	3 p. m.	63	67
4 a. m.	34	43	4 p. m.	62	67
5 a. m.	34	43	5 p. m.	62	65
6 a. m.	35	42	6 p. m.	61	62
7 a. m.	37	43	7 p. m.	59	57
8 a. m.	42	45	8 p. m.	56	55
9 a. m.	46	51	9 p. m.	53	55
10 a. m.	49	55	10 p. m.	50	55
11 a. m.	54	60	11 p. m.	47	52
12 a. m.	56	62	12 p. m.	45	51

21. Precipitation Charts. In Fig. 16 we have represented diagrammatically, by the usual method, the mean monthly precipitation at Tacoma, Washington; Madison, Wisconsin; Tampa, Florida; and Yuma, Arizona. In these charts the amount of precipitation is represented by the height of a shaded rectangle. This method of representation is similar to that used in Fig. 14 and enables one to see at a glance when the rainy and the dry periods occur at these localities.

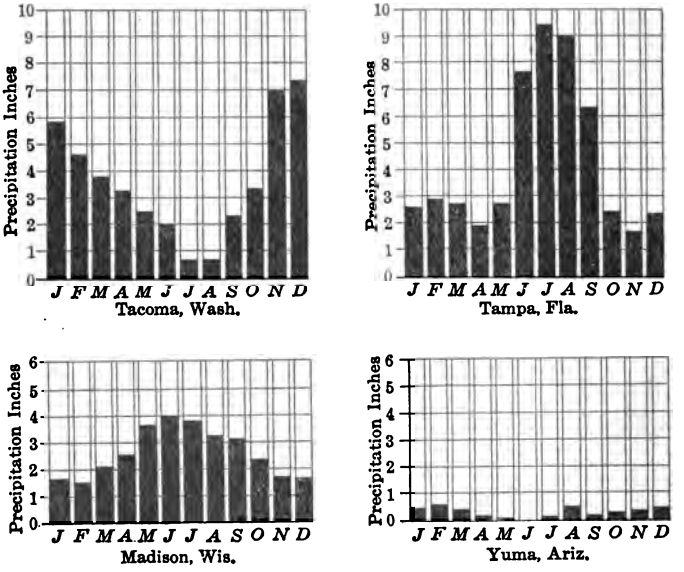


FIG. 16.—Precipitation charts.

22. Marking Axes. Whenever data are represented upon squared paper, both horizontal and vertical axes should be marked in such a way as to make perfectly clear what quantities are represented and what length represents a unit of measure of that quantity. If two or more curves are drawn upon the same set of axes, they should be so marked as to show definitely what each curve represents.

Occasions may arise when it is desirable to draw upon the same

sheet of paper, for comparison, two curves having in common only one element, such as time. A scheme for doing this is illustrated by exercise 12, below.

The length chosen to represent a unit is entirely arbitrary and need not be the same along both axes. This is already illustrated in Fig. 15. Further, the numbering in the horizontal and vertical directions need not be placed upon the axes, as will be illustrated by exercise 12, below.

Exercises

1. At what hour was the temperature a maximum on May 14, 1910? At what hour was the temperature a maximum on Oct. 10, 1910?

2. At what time was the temperature a minimum on May 14, 1910? At what time was the temperature a minimum on Oct. 10, 1910?

3. At what hour was the temperature increasing the fastest? At what hour was the temperature decreasing the fastest?

4. During what hours of the day was the temperature decreasing? During what hours of the day was the temperature increasing?

5. What was the temperature at 10:30 a. m.? 10:30 p. m.?

6. What was the temperature at 6:15 a. m.? 6:15 p. m.?

7. At what time was the temperature on May 14 the same as on October 10?

8. For what portion of the day was it warmer on May 14 than on October 10?

9. During what hours was the temperature below the freezing point?

10. Plot data given in Table III.

11. Plot data given in Table IV.

12. Plot the data given in Table V upon squared paper. Use the same horizontal axis for all three curves. Put the temperature curves above the discharge curve, using the same horizontal (time) scale for both. Let 1 cm. (or 1/2 inch) on the vertical scale represent 0.1 second-foot[1] discharge, and 10° temperature. Start the temperature scale with 60°, and place the 60, 6 cm. (or 3 inches) above the horizontal scale. Start the discharge scale with 3.4 placed on the horizontal scale.

[1] Second-foot (or sec-ft.), when applied to the measurement of flow of water means one cubic foot per second.

TABLE III.—MAXIMUM AND MINIMUM TEMPERATURES FOR OCTOBER, 1911, AT MADISON, WISCONSIN

Date	Max. temp., ° F.	Min. temp., ° F.	Date	Max. temp., ° F.	Min. temp., ° F.
1	55	50	17	61	51
2	56	50	18	67	47
3	72	50	19	57	45
4	61	48	20	54	39
5	51	42	21	46	38
6	63	47	22	48	35
7	53	45	23	42	34
8	61	40	24	46	30
9	62	45	25	47	37
10	61	49	26	38	30
11	65	48	27	42	28
12	69	49	28	43	29
13	57	47	29	53	30
14	61	50	30	45	37
15	57	49	31	40	32
16	65	52			

TABLE IV.—HOURLY TEMPERATURES AT MADISON, WISCONSIN, FOR APRIL 11, 1911, AND NOVEMBER 10, 1911

Hour	Temp., ° F., Nov. 10, 1911	Temp., ° F., April 11, 1911	Hour	Temp., ° F., Nov. 10, 1911	Temp., ° F., April 11, 1911
a. m.			p. m.		
1:00	59	39	1:00	62	59
2:00	59	40	2:00	60	61
3:00	59	40	3:00	59	59
4:00	59	40	4:00	51	58
5:00	60	40	5:00	44	56
6:00	60	40	6:00	39	53
7:00	61	41	7:00	35	53
8:00	62	44	8:00	31	52
9:00	64	47	9:00	28	51
10:00	66	50	10:00	25	47
11:00	68	55	11:00	22	46
12:00	70	57	12:00	20	45

13. What was the discharge at 7 : 00 p. m.? At 10 : 00 a. m.?

14. What was the minimum rate of discharge? At what time?

15. What was the water temperature when the discharge was a minimum?

16. What was the maximum water temperature? At what time?

17. What was the maximum air temperature? At what time?

18. What was the greatest difference between the air temperature and the water temperature?

19. At what hour did this "maximum" difference occur?

20. What was the minimum difference between the air temperature and the water temperature?

21. At what hour did this "minimum" difference occur?

22. Can you think of any cause which would produce the indenture in the air temperature curve at 1:30 p. m., at which time a normal temperature curve should have a maximum?

23. Can you think of any reason why the discharge curve should have a minimum when the temperature curves have maxima?

TABLE V.—DISCHARGE OF A SEEPAGE DITCH

Time, Aug. 24, 1905	Discharge of ditch, sec.-ft.	Temp. of water, ° F.	Temp. of air, ° F.
8 : 00 a. m.	3.72	65	.....
9 : 00 a. m.	3.70	70	74
11 : 00 a. m.	3.66	79	84
1 : 30 p. m.	3.52	83	85
2 : 15 p. m.	3.49	.....	.....
3 : 30 p. m.	3.52	84	90
5 : 30 p. m.	3.66	78	88
6 : 00 p. m.	3.73	.....	.....
8 : 00 p. m.	3.84	67	76

23. The Point. The location of a point upon a sheet of rectangular coördinate paper is fixed by giving its distances from the vertical and horizontal axes. The distance from the vertical axis is called the **abscissa,** and the distance from the horizontal axis is called the **ordinate** of the point. These two distances are called the **coördinates of the point.** The horizontal axis is commonly called the **X-axis,** or the axis of x; and the vertical axis, the **Y-axis,** or the axis of y. The abscissa and ordinate are then

called, respectively, the **x and y coördinates.** When writing the coördinates of a point, the value of the abscissa is placed before the value of the ordinate, and separated from it by a comma. Thus, "the point (3, 4)," (read, "three, four"), is a point three units to the right of the Y-axis, and four units above the X-axis. The point $(-2, 6)$ is two units to the left of the Y-axis and six

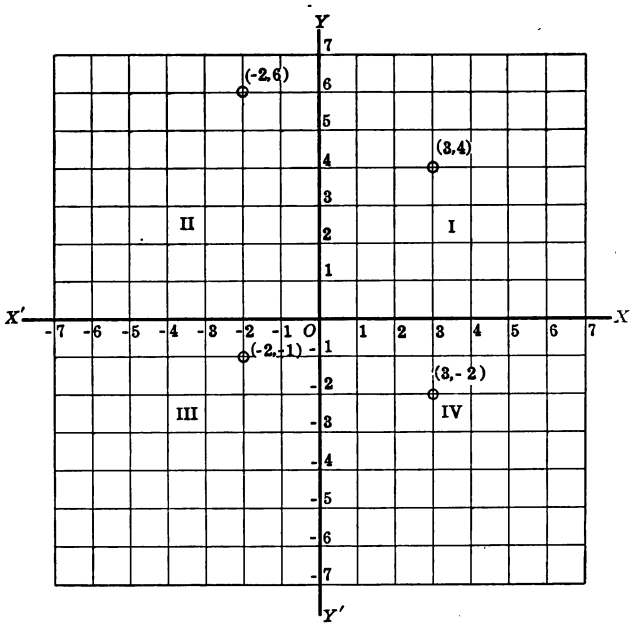


FIG. 17.

units above the X-axis. The point $(-2, -1)$ is two units to the left of the Y-axis and one unit below the X-axis. The point $(3, -2)$ is three units to the right of the Y-axis and two units below the X-axis. These four points are shown on the sheet of squared paper in Fig. 17.

The two axes divide the plane into four parts, called **quadrants.** They are numbered from "one" to "four" as shown in Fig. 17.

Exercises

1. Plot the following points: (*a*) (1, 1); (*b*) (2, 3); (*c*) (− 3, 2); (*d*) (− 1, − 1); (*e*) (− 2, − 3); (*f*) (3, − 4); (*g*) ($\sqrt{2}$, 1); (*h*) (− $\sqrt{3}$, $\sqrt{5}$).

2. Find the distance of each of the following points from the origin: (*a*) (1, 1); (*b*) (− $\sqrt{2}$, $\sqrt{2}$); (*c*) (− 3, − 2).

3. Find the distances between the following pairs of points: (*a*) (1, 1) and (2, 2); (*b*) (− 2, 3) and (− 5, − 6); (*c*) (3, − 5) and (− 1, 0).

4. What is the y-coördinate of any point upon the axis of x?

5. What is the x-coördinate of any point upon the axis of y?

6. What is the y-coördinate of any point upon a straight line parallel to the axis of x, three units above it?

7. What is the x-coördinate of any point upon a straight line parallel to the Y-axis, five units to its left?

8. What relation exists between the coördinates of any (*every*) point of a line bisecting the angle between the positive directions of the two axes? Between the negative directions of the two axes?

9. What relation exists between the coördinates of any (*every*) point of a line bisecting the angles having the positive direction of the Y-axis and the negative direction of the X-axis as sides?

10. What relation would exist between the x- and y-coördinates of any (*every*) point of the line in exercise 9, if it were raised five units parallel to itself? If it were lowered six units ?

11. What relation would exist between the coördinates of any (*every*) point of the line in exercise 9, if the line were moved up three units parallel to itself? If it were lowered five units?

12. What are the coördinates of the origin?

13. Compute for every 10° temperature (between 0° and 200°) read upon a Fahrenheit scale, the corresponding reading upon a centigrade scale. Put your results in tabular form, using a sheet of paper, form *M*7.

14. Compute for every 10° temperature (between − 20° and 100°) read upon a centigrade scale, the corresponding readings upon the Fahrenheit scale. Put your results in tabular form, using a sheet of paper, form *M*7.

15. From the table constructed in exercise 13, convert the following readings Fahrenheit into readings centigrade: (*a*) 12.6; (*b*) 21.7; (*e*) 28.55; (*d*) 116.75; (*e*) 151.86.

16. From the table constructed in exercise 14, convert the following

readings centigrade into readings Fahrenheit: (*a*) -2; (*b*) 22.2; (*c*) 61.3; (*d*) 62.6; (*e*) 75.4.

17. If x represents the number of degrees temperature, as read upon a Fahrenheit scale, and if y represents the number of degrees of the same temperature as read upon a centigrade scale, give the equation connecting y and x.

18. Plot upon squared paper the numbers in the tables constructed in exercises 13 and 14. Plot degrees Fahrenheit along the horizontal axis and degrees centigrade along the vertical axis. Recompute numbers, if any, whose plotted points do not fall upon a straight line connecting the points (32, 0) and (212, 100).

19. Construct a table, for every sixty-fourth of an inch, converting sixty-fourths into decimal equivalents. Compute to three decimal places.

20. Plot the corresponding numbers found in exercise 19 upon squared paper. Do all the points plotted fall upon a straight line passing through the origin?

21. Construct a table converting centimeters, from 0 to 100, into inches. One meter = 39.37079 inches. Compute to three decimal places. Plot the numbers of the table upon squared paper. Do all the points fall upon a straight line passing through the origin?

24. The Double Scale. From the preceding exercises we have seen that relations between numbers may be represented by tables, by equations, and by curves upon squared paper. A fourth method of representing a relation between numbers will be explained by means of an illustration. If x represents a pressure read in feet of water, and if y represents the same pressure read in inches of mercury (since mercury is 13.55 times as heavy as water), the relation between x and y is

$$y = \frac{12x}{13.55},$$

or

$$y = 0.886\,x.$$

In Fig. 18, the scale on the under side of the horizontal line represents pressure measured in feet of water, the scale on the upper side represents pressure measured in inches of mercury. The scales are so drawn that readings directly opposite the horizontal line represent the same pressure, *i.e.*, the length representing a unit

on the upper scale is 1.13 times as long as the length representing a unit on the lower scale. From this double scale, one can easily convert pressure readings in feet of water into inches of mercury, or *vice versa*.

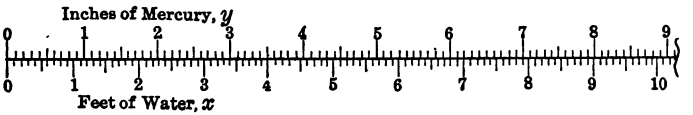


FIG. 18.—Double scale showing relation between pressure measured in inches of mercury and in feet of water.

Exercises

1. Draw a double scale connecting centimeters and inches.
2. Draw a double scale connecting inches and tenths of a foot.
3. Draw a double scale connecting pressure measurements in feet of water and pounds per square inch. A cubic foot of water weighs 62.5 pounds.

25. The Equation of a Curve, the Graph of an Equation. Any (*every*) point of a line parallel to the axis of x and three units above it has an ordinate three, and no point whose ordinate is three is found off the line. Since the equation $y = 3$ is satisfied by the coördinates of every point of this line and by the coördinates of no other point, it is said to be the equation of the line. The equation of a line parallel to the X-axis and ten units below it is $y = -10$. The equation of the X-axis is $y = 0$. The equation of a line parallel to the Y-axis, five units to its right, is $x = 5$.

The equation of a curve is an equation satisfied by the coördinates of every point of the curve and by the coördinates of no other point.

The graph of an equation is the locus of a point whose coordinates satisfy the equation. To illustrate: if we have a circle,[1] radius 5 and center at the origin of coördinates, its equation is $x^2 + y^2 = 5^2$. For, if we draw from any point of this circle a line to the origin and a perpendicular to the X-axis we have formed a right triangle whose legs are x and y, and whose hypotenuse is 5;

[1] The term "circle" as used in this book means the locus of a point at a constant distance from a fixed point.

and if the point is chosen off the circle, the hypotenuse of the right triangle is no longer 5. Thus, for points of the circle we have $x^2 + y^2 = 25$, and for points off of the circle, $x^2 + y^2 \neq 25$.[1]

The graph of the equation $x^2 + y^2 = 25$ is a circle, radius 5 and center at the origin.

The equation of a straight line passing through the origin and making an angle of 45° what the positive direction of the axis of x is $y = x$. The graph of the equation $y = x$ is a straight line passing through the origin, making an angle of 45° with the positive direction of the axis of x.

For brevity, instead of saying "the circle," or "the straight line," or "the curve"—naming the equation—"representing the equation"—we shall simply say the circle, or the straight line, or the curve—naming the curve. Thus, the circle $x^2 + y^2 = 25$; the straight line $y = x$, and the curve $x^3 - 3xy - y^2 = 1$.

Exercises

1. What is the equation of the axis of x?

2. What is the equation of the axis of y?

3. What is the equation of a line parallel to the X-axis, three units above? Five units above? One hundred units below? Five units below? Fifty units below?

4. What is the equation of a line parallel to the axis of y, five units to the right? Ten units to the right? Six units to the left? Eighty units to the left?

5. What is the equation of a right line passing through the origin and bisecting the angle between the positive directions of the coördinate axes?

6. What is the equation of a straight line passing through the origin and bisecting the angle between the positive direction of the axis of y and the negative direction of the axis of x?

7. What is the graph of $y = x$? Of $y = -x$? Of $y = x + 2$? Of $y = x - 3$?

8. What is the graph of $y = -x$? Of $y = -x + 1$? Of $y = -x + 3$? Of $y = -x - 6$?

9. What is the equation of a straight line making an angle of 45° with the positive direction of the axis of x and cutting the Y-axis five units above the origin? Ten units above the origin? Six units below the origin? Twenty units below the origin?

[1] "$\neq$" is read "is not equal to," or "is different from."

10. What is the equation of a straight line making an angle of 135° with the positive direction of the axis of x and cutting the Y-axis one unit above the origin? Three units above the origin? Five units below the origin?

11. What is the equation of the circle, center at the origin and radius 1? Radius 2? Radius 3?

12. Show that the equation of a circle, radius 5 and center at the point (2, 3), is $(x-2)^2+(y-3)^2=5^2$, or $x^2+y^2-4x-6y=12$.

HINT: Take any point, P, of the circle. Its coördinates are x and y. Draw PA perpendicular to the axis of x. Call the center of the circle C, and draw CB perpendicular to PA. Draw PC. PCB is a right triangle, and

$$CB^2+BP^2=CP^2,$$

or

$$(x-2)^2+(y-3)^2=5^2,$$

or

$$x^2-4x+4+y^2-6y+9=25,$$

or

$$x^2+y^2-4x-6y=12.$$

The student will show that the same equation is obtained when the point P is to the right of, and below C; to the left of the Y-axis; below the X-axis. When the point P is to the left of the Y-axis the student must remember that x is a negative number; when below the x-axis the y is a negative number.

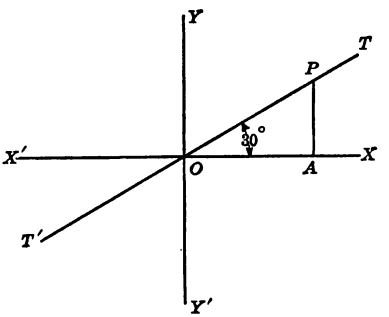


FIG. 19.

13. Show that the equation of a circle, radius 5, center at the point (3, 2), is $x^2+y^2-6x-4y=12$.

14. Show that the equation of a circle, radius 5, center at the point $(-2, 3)$, is $x^2+y^2+4x-6y=12$.

15. Show that the equation of a circle, radius 5, center at the point $(-3, 2)$, is $x^2+y^2+6x-4y=12$.

16. Find the equation of a straight line passing through the origin and making an angle of 30° with the positive direction of the X-axis. Let $T'OT$, Fig. 19, be the line. Let $P=(x, y)$ be any (*every*) point of

the line. Let PA be the ordinate, y; and let OA be the abscissa, x. OAP is a 30° triangle, and (see exercise 3, page 8)

$$\frac{AP}{OA} = \frac{y}{x} = \frac{1}{\sqrt{3}},$$

or

$$y = \frac{x}{\sqrt{3}}$$

is the equation of the line. For, the coördinates of every point of the line satisfy the equation and the coördinates of any point not of the line do not satisfy the equation. If the line were raised three units, its equation would be $y = \frac{x}{\sqrt{3}} + 3$; if it were lowered ten units, its equation would be $y = \frac{x}{\sqrt{3}} - 10$.

17. Show that the equation of a right line passing through the origin and making an angle of 60° with the positive direction of the axis of x is $y = \sqrt{3}x$. Show that if this line were raised six units its equation would be $y = \sqrt{3}x + 6$; and if it were lowered five units, its equation would be $y = \sqrt{3}x - 5$.

18. Find the equation of a straight line cutting the Y-axis one unit above the origin and the X-axis one unit to the left of the origin; cutting the Y-axis one unit above the origin and the X-axis one unit to the right of the origin; cutting the Y-axis one unit below the origin and the X-axis one unit to the right of the origin; cutting the Y-axis one unit below the origin and the X-axis one unit to the left of the origin.

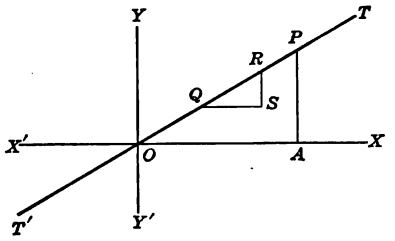


FIG. 20.

26. The Equation of the Straight Line. Let $T'OT$, Fig. 20, be *any* straight line passing through the origin. If, in addition to stating that the line passes through the origin, its direction is given, the line is fixed. Its direction may be given by the angle XOT. Let Q and R be *any* two points upon the line. Suppose a describing point slide along the line from Q to R, its x-coördinate

will change by the amount QS and its y-coördinate will change by the amount SR. If the line extends in the first and third quadrants, both QS and SR will be positive, or both negative, depending upon whether R is above and to the right of Q, or below and to the left of Q. If the line extends in the second and fourth quadrants, QS and SR will have opposite signs. If R is above and to the left of Q, QS will be negative, and SR positive; if R is below and to the right of Q, QS will be positive and SR negative. It is readily seen, however, that for any fixed line the ratio $\frac{SR}{QS}$ is constant (remains unchanged as the positions of Q and R are changed). Let us abbreviate this ratio, $\frac{SR}{QS}$, by a. If the line extends into the first and third quadrants, a is positive; if into the second and fourth quadrants, a is negative. If the line is very nearly horizontal, a is numerically small. If the line is very nearly vertical, a is numerically very large. For a line making an angle of 30° with the positive direction of the axis of x, a is $1/\sqrt{3}$. For a line making an angle of 45° with the positive direction of the axis of x, a is 1. For a line making an angle of 60° with the positive direction of the axis of x, a is $\sqrt{3}$. For lines making angles of 120°, 135°, and 150° with the positive direction of the axis of x, a is, respectively, $-\sqrt{3}$, -1, and $-\frac{1}{\sqrt{3}}$.

If the line were to rotate counter-clockwise about the origin from a horizontal position toward a vertical position, a would increase continuously from zero through all positive real values. If the line were to rotate clockwise from a horizontal position toward a vertical position, a would decrease continuously from zero through all negative real values. From this, we see that every line passing through the origin has an a, and for every value assigned to a there corresponds one and only one straight line through the origin.

The direction of a line is fixed by the algebraic value of a as well as by the number of degrees in the angle between the line and the positive direction of the axis of x. The numerical value of a is a measure of the steepness of the line. It represents the number

of times faster a point moves in a vertical direction than in a horizontal direction as it slides along the line. a is called the **slope of the line.** The term slope is analogous to the term percent grade in railroad work. A 1 percent grade means a rise of 1 foot for every 100 feet of horizontal distance, or a slope of 1/100. Again we speak of the hydraulic gradient of a stream as 2 feet per 1000, or as 8 feet per mile; these mean, respectively, a slope equal to 2/1000 and 8/5280.

Let P, Fig. 20, be *any* point on the line. Then $\frac{AP}{OA} = a$, or $y/x = a$, or $y = ax$. This, then, is the equation of a straight line *passing through the origin.*[1] If the line is raised two units, its equation becomes $y = ax + 2$; if raised five units, $y = ax + 5$; if lowered ten units, $y = ax - 10$. The equation of a line parallel to the line $y = ax$, and cutting the Y-axis b units from the origin, is $y = ax + b$. If the line cuts the Y-axis above the origin, b is positive. If the line cuts the Y-axis below the origin, b is negative.

$y = ax + b$ is called the **slope equation** of the straight line, a is called the **slope,** and b the **y-intercept.**

Exercises

1. Give the slope and y-intercept for each of the following lines:

(*a*) $y = 2x + 1$;	(*g*) $y = -3x + 2$;	(*m*) $y = \frac{1}{2}x + 3$;
(*b*) $y = 2x - 3$;	(*h*) $y = x + 1$;	(*n*) $y = \frac{1}{2}x - 3$;
(*c*) $y = 3x - 6$;	(*i*) $y = x - 1$;	(*o*) $y = -\frac{1}{2}x$;
(*d*) $y = 2x - 1$;	(*j*) $y = x + 2$;	(*p*) $y = -\frac{3}{4}x - 8$;
(*e*) $y = -2x - 2$;	(*k*) $y = -x + 6$;	(*q*) $y = -\frac{3}{4}x$;
(*f*) $y = -3x + 4$;	(*l*) $y = -x + 2$;	(*r*) $y = -0.01x - 6$.

2. In exercise 1, what lines are parallel to (*a*)? To (*c*)? To (*e*)? To (*f*)? To (*h*)? To (*k*)? To (*m*)? To (*p*)?

3. If a point were to move along the lines of exercise 1, so that its x-coördinate is decreased by one unit, would the y-coördinate be increased or decreased and how much?

4. Find the equation of a straight line passing through the points P, (3, 2), and Q, (1, 5). To solve the problem is to find the value of a and of b and substitute in the equation $y = ax + b$. As we pass from the point P to the point Q, the x-coördinate decreases two units

[1] The student must guard against getting the idea that the slope of a line is, in general, y over x. This is *only* true when the line passes *through the origin.*

and the y-coördinate increases three units. Then the slope, a, of the line is $-3/2$. Let R be the point where the line through P and Q cuts the Y-axis. Draw PT perpendicular to the Y-axis, meeting it at the point T. Draw QS perpendicular to PT, meeting it at S. PSQ and PTR are similar triangles, and we have

$$\frac{TP}{SP} = \frac{TR}{SQ},$$

or

$$\frac{3}{2} = \frac{TR}{3},$$

or

$$TR = \frac{9}{2}$$

and $b = 1\frac{1}{2}$, since $b = OT + TR$. The equation of the line is

$$y = -\frac{3}{2}x + \frac{13}{2},$$

or

$$3x + 2y = 13.$$

This work can be checked, for, substituting for x and y the coördinates of the points P and Q, the equation is satisfied in each case.

5. Does the line of exercise 4 pass through the point (2, 7/2)? The answer is yes, for substituting 2 for x and 7/2 for y, in the equation of the line, the equation is satisfied.

6. Find the equation of the straight line passing through the points (*a*) (− 3, 2) and (2, 1); (*b*) (1, 1) and (2, 2); (*c*) (− 1, 3) and (2, − 3); (*d*) (− 3, −1) and (−1, 3); (*e*) (− 1, 2) and (6, − 1).

7. Which lines of exercise 6 pass through the point (− 2, 6)?

8. Find the equation of a straight line passing through the origin and the point (3, 5); through the points (3, − 1) and (− 3, − 1); through the points (− 3, 5) and (− 3, 2).

27. The Graph of an Equation of the First Degree between x and y. In § **26** it was shown that every equation of the form $y = ax$ represents a straight line passing through the origin; and further, when the line was moved up or down its equation became $y = ax + b$, where b represents the algebraic distance the line was translated in the y direction. If the line were moved from a very low position to a very high position, the b would take on all real values between its very small initial value and its very large terminal value. Thus, by moving the line over a greater and greater

vertical distance, b can be made to take any real value whatever. Any equation then of the form $y = ax + b$, for any values given to a and b, represents a straight line. It is readily seen that this form of the equation includes all lines excepting those parallel to the Y-axis. The equation of a line parallel to the Y-axis is, however, $x = k$, where the constant k is the number of units the line is distant from the Y-axis. The equation of any straight line is then either of the form $y = ax + b$ or of the form $x = k$. Since any equation of the first degree between x and y, $Ax + By = C$, may be put in one of these forms, it is the equation of a straight line, and is called a **linear equation.** To illustrate: $3x + 2y + 6 = 0$ may be written $y = -\frac{3}{2}x - 3$, the equation of a straight line having the slope $-\frac{3}{2}$, and a y-intercept of -3.

Exercises

Find the slope and intercept of each of the following lines:

1. $2x - 3y = 1$.
2. $3x - 2y - 6 = 0$.
3. $x - y = 1$.
4. $3x - 2y + 6 = 0$.
5. $y - 3x = 5$.
6. $y + x = 0$.

7. Find the point of intersection of each line given above with the axis of x. (The distance from the origin to the point of intersection of a line with the X-axis is called the x-intercept.)

HINT: This is done by putting y equal to zero, and solving the resulting equation for x. Why does this give the x-intercept?

8. Draw the straight lines of exercises 1 to 6 upon one sheet of squared paper. Use 1 inch (or 2 cm.) to represent the unit and place the origin near the center of the sheet.

9. Find the coördinates of all points of intersection of these lines in so far as they intersect upon the sheet of squared paper.

The coördinates of the points of intersection are to be read to two decimal places. To make this possible great care must be exercised in drawing the lines. Tabulate your results. In the first column place the numbers of the lines, as 2–5, meaning the line in exercise 2, and the line in exercise 5. In the second column place the x-coördinate and in the third column place the y-coördinate of the point of intersection of the two lines. A fourth and a fifth column are to be filled in with the results obtained from the next exercise.

10. Consider all pairs of equations given in exercises 1 to 6 as simultaneous equations and solve for x and y. Record the results in the table of the preceding exercise. Compare the corresponding x

values and the corresponding y values found by the two methods. Should the corresponding values be the same? Why?

28. Analytic Method of Finding a and b. In exercise 4, § 26, there was described a geometric method for finding the a and the b in the equation of a straight line passing through two points. An analytic method of finding a and b will now be illustrated. Let (3, 2) and (1, 5) be the points through which the line passes. The coördinates of each point when substituted for x and y in the general equation of the line, $y = ax + b$, must satisfy that equation. Thus we have, upon substitution, $2 = 3a + b$, and $5 = a + b$. These two equations are called conditional equations, for they are, in mathematical language, statements of the two and only two conditions imposed upon the line.[1] The first equation is the equation of condition which says that the point (3, 2) is upon the line $y = ax + b$; the second equation is the equation of condition which says that the point (1, 5) is upon the line $y = ax + b$. Solving these two conditional equations for a and b we obtain $a = -\frac{3}{2}$, and $b = \frac{13}{2}$. Substituting these values in $y = ax + b$, the equation of the line through the two given points becomes $y = -\frac{3}{2}x + \frac{13}{2}$, or $3x + 2y = 13$.

Exercises

1. Find, by the analytic method, the equation of the line through each of the following pairs of points:

(*a*) (1, 2) and (− 3, 6); (*c*) (0, 0) and (− 1, − 1);

(*b*) (− 2, 6) and (3, − 2); (*d*) (0, 0) and (− 2, − 3).

2. Plot upon squared paper the data given in Table VI. Plot hydraulic gradient[2] along the Y-axis using 1 cm. (or 1/2 inch)

[1] A conditional equation may be defined as an equation expressing some condition given in the problem. To illustrate: If two numbers x and y are to be found such that their sum is 9, and the sum of their squares is 25, we have

$$x + y = 9$$

and

$$x^2 + y^2 = 25.$$

The first equation expresses the condition imposed upon x and y, that their sum is 9. The second equation expresses the condition that the sum of the squares of the numbers is 25.

[2] By hydraulic gradient is meant the drop of the surface of water in moving down stream a given distance. Thus in the table the hydraulic gradient of 10 feet per thousand feet means a drop of the water plane (water surface) at the rate of 10 feet for every 1000 feet measured along the horizontal in the direction of greatest decline of the plane.

to represent 10 feet. Plot velocity along the X-axis, using 2 cm. (or 1 inch) to represent 10 feet. From the plotted points it appears as if the points would fall (or would nearly fall) along a straight line if there were no errors of observation, or if all the readings were taken under the same conditions. The best that can be done in this case is to assume a linear relation between hydraulic gradient and velocity. Draw with the aid of a transparent triangle a straight line among the points which in your judgment is the best line to represent this relation. Does the line pass through the origin? Compute a by taking two points at or near the ends of the line. The student must remember that a is the ratio of the number of units in the change in the y-coördinate to the number of units in the change in the x-coördinate.

3. Plot upon squared paper the data for the 8-inch well given in Table VII. Plot head[1] along the Y-axis using 1 cm. (or 1/2 inch) to represent 1 foot. Plot yield along the X-axis, using 1 cm. (or 1/2 inch) to represent 10 gallons. As in the preceding exercise draw a straight line among the points and determine the a.

TABLE VI.—VELOCITY OF WATER THROUGH SAND

Hydraulic gradient, feet per 1000	Velocity of water, feet per 24 hours	Hydraulic gradient, feet per 1000	Velocity of water, feet per 24 hours
31.9	16.9	21.0	10.1
20.8	11.4	119.1	58.8
54.1	24.5	55.8	22.7
10.0	4.3	.....	.....

TABLE VII.—YIELD OF AN 8-INCH AND OF A 14-INCH WELL

Size of well, inches	Head, feet	Yield, gallons per minute	Size of well, inches	Head, feet	Yield, gallons per minute
14	2.02	31.4	8	2.50	39.5
14	4.00	59.7	8	3.85	49.4
14	6.40	93.2	8	6.47	89.3
14	8.68	121.1	8	8.28	116.3
14	11.73	151.1	.....	.....	.....

[1] By "head" in this problem is meant the distance the water was lowered in the well below its normal position. Thus, a head of 2.5 feet in the table means that when 39.5 gallons of water per minute were pumped from the well the elevation of the surface of the water in the well was 2.5 feet below its normal elevation.

4. Plot upon squared paper the data for the 14-inch well given in Table VII. Draw a straight line among the points and find its a.[1]

29. Cost of Concrete. Suppose concrete mixture is made by mixing cement, sand, and crushed rock. The expression "mixture (m, n, b)" means the relative amounts, by volume, of the three materials. Thus, a mixture (1, 2, 3) means that for each volume of cement there are two volumes of sand and three volumes of rock.

Curves will now be drawn from which the cost of a cubic yard of concrete may be determined. The method of drawing the curves will be illustrated by an example. The student is to construct the drawing as explained. Suppose the percent voids of rock is 45 and that of sand 33. Let us take the mixture (1, 2, 3). For each cubic yard of cement, 2 cubic yards of sand and 3 cubic yards of rock are used. In the 3 cubic yards of rock there are $3 \cdot 0.45 = 1.35$ cubic yards of voids. Then 1.35 cubic yards of the 2 cubic yards of sand will be taken up by the voids of the rock leaving 0.65 cubic yard. Thus mixing 3 cubic yards of rock and 2 cubic yards of sand will give but $3 + 0.65 = 3.65$ cubic yards of mixture. In the 2 cubic yards of sand there are $2 \cdot 0.33 = 0.66$ cubic yard of voids. Thus there is an excess of $1 - 0.66 = 0.34$ cubic yard of cement above filling the voids of the sand. Hence in mixing the rock, sand, and cement there will be $3.65 + 0.34 = 3.99$ cubic yards of mixture.

If x_1 is the cost (say in dollars) of a cubic yard of stone, the cost, y_1, of the stone in 1 cubic yard of concrete is

$$y_1 = \frac{3}{3.99} x_1$$

or

$$y_1 = 0.752\, x_1 \qquad (1)$$

Upon a sheet of squared paper draw a straight line representing equation (1). Let 10 cm. represent \$1. Then each centimeter will represent 10 cents. To draw this line join the points (0, 0) and (2, 1.504).

[1] The equations found in exercises 2, 3, and 4 very probably do not represent the true relations among the measured numbers, and are at best only approximate relations. Such equations are called empirical equations, or empirical laws.

If x_2 represents the cost of 1 cubic yard of sand, the cost, y_2, of the sand in 1 cubic yard of concrete is

$$y_2 = \frac{2}{3.99} x_2$$

or

$$y_2 = 0.501\, x_2 \qquad (2)$$

Plot equation (2) upon the same sheet of paper upon which equation (1) was plotted, using the same scales.

A barrel (four sacks) of cement contains $3\frac{3}{4}$ cubic feet, or 5/36 cubic yard. If x_3 represents the cost of cement per barrel, the cost, y_3, of cement in 1 cubic yard of concrete is

$$y_3 = \frac{36}{5 \cdot 3.99} x_3$$

or

$$y_3 = 1.805\, x_3 \qquad (3)$$

Draw this line upon the same sheet of paper upon which equations (1) and (2) were drawn.

Mark the first line drawn "rock;" the second "sand;" and the third "cement." From these three lines the cost of material in 1 cubic yard of concrete can be scaled off, if the costs of stone, sand, and cement are given. Thus, suppose stone worth \$1.40 per cubic yard, sand \$1.25 per cubic yard and cement \$1.10 per barrel. From the point 1.4 on the X-axis go up to the rock line, thence to the left to the Y-axis and obtain the reading \$1.05, the cost of the stone in 1 cubic yard of concrete. From the point 1.25 on the X-axis go up to the sand line, thence to the left to the Y-axis and obtain the reading \$0.63, the cost of the sand in 1 cubic yard of concrete. From the point 1.1 on the X-axis go up to the cement line, thence to the left to the Y-axis and obtain \$1.99, the cost of the cement in 1 cubic yard of concrete. The total cost of material in 1 cubic yard of concrete is then \$1.05 + \$0.63 + \$1.99 = \$3.67.

The following table gives the number of cubic yards of rock, cubic yards of sand, and barrels of cement in 1 cubic yard of concrete based on 45 percent voids in rock and 33 percent voids in sand.

Mixture	Rock, cu. yds.	Sand, cu. yds.	Cement, barrels
1:2 :3	0.752	0.501	1.805
1:2 :4	0.881	0.446	1.586
1:2½:5	0.922	0.461	1.327
1:3 :5	0.951	0.475	1.125

Exercises

1. Verify the above table.

2. Construct curves from which to scale off the cost of material used in a 1, 3, 5 mixture of concrete.

3. From the drawing of exercise 2, find the cost of the material in 1 cubic yard of concrete, if rock is worth \$1.50 a cubic yard, sand \$1.10 a cubic yard and cement \$1.30 per barrel.

30. The Graph of $y = px^2$.

Exercises

1. Upon a sheet of paper, form *M7*, construct a table per the following instructions. Head the first vertical column x, the second x^2, the third $\frac{1}{2}x$, the fourth $\frac{1}{3}x^2$, and the fifth $\frac{1}{4}x^2$. In the horizontal spaces in the first column place the numbers 0, 0.2, 0.4, 0.6, 0.8, 1.0, 1.2, . . ., 4.6, 4.8, 5.0.[1] In the second column place the squares of these numbers, as 0, 0.04, 0.16, 0.36, 0.64, 1.0, . . ., 21.16, 23.04, and 25.00. In the third, fourth and fifth columns place, respectively, the halves, the thirds and the fourths of the numbers in the second column. Head this table "Table of Squares."

2. Plot upon squared paper a curve for $y = x^2$. Let the X-axis run the shorter and the Y-axis the longer dimension of the sheet of paper. Choose the origin near the lower center of the page. Plot points for negative values of x as well as for positive values. Make use of your "Table of Squares." Let 2 cm. (or 1 inch) on both axes represent one unit. Use a very sharp pointed pencil in plotting these points, and exercise great care in locating them as accurately as possible. Do not draw a curve through the plotted points until all exercises of this list have been completed.

3. Plot a curve for $y = \frac{1}{2}x^2$. Follow directions given in exercise 2.

4. Plot a curve for $y = \frac{1}{3}x^2$. Follow directions given in exercise 2.

5. Plot a curve for $y = \frac{1}{4}x^2$. Follow directions given in exercise 2.

6. Plot a curve for $y = \frac{1}{5}x^2$. Follow directions given in exercise 2.

[1] The marks ". . ." mean "and so on," indicating all corresponding intermediate numbers.

7. From the coördinates of the points of the curve $y = x^2$ compute the slope of lines passing through the consecutive plotted points. The slope of the secant lines will be approximately the slope of the tangent lines drawn to the curve midway between the points. Thus. if we find a slope of 4.2 for the secant line drawn through the points whose x-coördinates are 2.0 and 2.2, we shall have the approximate slope of the tangent line drawn to the curve at the point whose x-coördinate is 2.1 Tabulate your results.

8. Same as exercise 7 but use curve $y = \frac{1}{2}x^2$.

9. Same as exercise 7 but use curve $y = \frac{1}{3}x^2$.

10. Same as exercise 7 but use curve $y = \frac{1}{4}x^2$.

11. Same as exercise 7 but use curve $y = \frac{1}{5}x^2$.

12. Plot upon squared paper the results of exercise 8. Plot x along the horizontal axis, and slope, a, along the vertical axis. Draw a straight line through the origin and among the plotted points. Measure the slope of this line; call it A_1.

13. Same as exercise 12 but use results of exercise 9. Call the slope A_2.

14. Same as exercise 12 but use results of exercise 10. Call the slope A_3.

15. Same as exercise 12 but use results of exercise 11. Call the slope A_4.

16. Divide A_2 by $\frac{1}{2}$; A_3 by $\frac{1}{3}$; and A_4 by $\frac{1}{4}$. These quotients and A_1 should all be equal to 2.

It can be shown by the method of the Calculus that the slope of a tangent line drawn to the curve $y = px^2$ (p being a constant) is $2px$, where x is the abscissa of the point of tangency.[1]

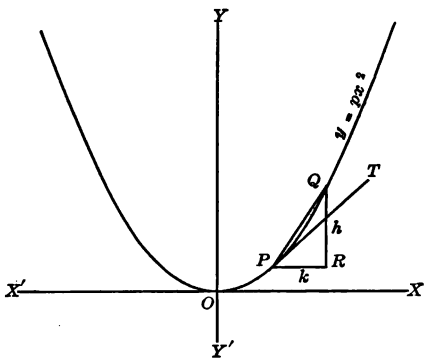


FIG. 21.

[1] A formal proof is given here. Let P, (x, y), Fig. 21, be *any* point, and Q *any* second point on the curve $y = px^2$. Let k, (PR), be the difference of the two x-coördinates, and let h, (RQ), be the difference of the two y-coördinates. The coördinates of Q are then $(x + k, y + h)$. For the point P we have $y = px^2$; and for the point Q we have $(y + h) = p(x + k)^2$. Subtracting, we have $h = 2pkx + pk^2$. Dividing by k we have $h/k = 2px + pk$. This shows that the slope, (h/k), of the secant line drawn through the points P and Q is

This is only saying that if a point move along the curve its upward motion (downward motion if the point is to the left of the y-axis) is at any instant $2px$ times its horizontal motion, or *its y-coördinate is changing 2px times as fast as its x-coördinate.* Here x represents the x-coördinate of the position of the moving point.

Experiment shows that a body falling freely from rest gives the law $s = 16.1t^2$, where s represents the distance, in feet, passed over in the time t, in seconds. If s be plotted along the vertical axis and t along the horizontal axis the slope of the tangent line drawn to the curve will be $32.2t$. Distance s, then, is changing $32.2t$ times as fast as t, *i.e.*, the speed of a body falling freely from rest is $32.2t$, when t is the time, measured in seconds, elapsed from the instant the body began to move.

If v represents speed, in feet per second, we have $v = 32.2t$. Plotting v along the vertical axis and t along the horizontal axis we obtain a straight line representing the relation between speed and time. The slope of his line is 32.2, *i.e.*, v changes 32.2 times as fast as t, or the acceleration of gravity is 32.2 feet per second per second.

17. Plot a curve for $y = x^2 - 2x + 1 = (x - 1)^2$. How is this curve related to the curve $y = x^2$?

18. Plot a curve for $y = x^2 - 2x + 2 = (x - 1)^2 + 1$. How is this curve related to the one plotted in exercise 17?

19. Plot a curve for $3y = x^2 - 4x + 13 = (x - 2)^2 + 9$, or $y = \frac{1}{3}(x - 2)^2 + 3$. How is this curve related to the curve for $y = \frac{1}{3}x^2$?

20. Without plotting, compare the following curves:

(*a*) $y = x^2$ and $y = -x^2$.

(*b*) $y = \frac{1}{2}x^2$ and $y = -\frac{1}{2}x^2$.

(*c*) $y = \frac{1}{3}x^2$ and $y = -\frac{1}{3}x^2$.

(*d*) $y = \frac{1}{4}x^2$ and $y = -\frac{1}{4}x^2$.

(*e*) $y = (x - 1)^2$ and $y = -(x - 1)^2$.

(*f*) $y = (x + 2)^2 + 3$ and $y = -(x + 2)^2 - 3$. $y = -(x + 2)^2 + 3$.

$2px + pk$. As the point Q slides along the curve and approaches P, k, (PR), grows smaller and smaller and approaches zero; the slope of the secant line then approaches $2px$. But, as Q approaches P the secant line rotates about the point P and approaches the tangent line PT. Therefore, the slope of the tangent line is the limit of the slope of the secant line, *i.e.*, $2px$.

21. Have the curves considered in exercise 20 lines of symmetry? Give the equation of the line of symmetry for each curve.[1]

22. Change each of the following equations to the form $y = p(x - q)^2 + r$, where p, q, and r are constants, and find the coördinates of the highest or lowest point on the curve:

(*a*) $4x^2 + 3y = 3x + 5,$

or $3y = -4x^2 + 3x + 5,$

or $-\frac{3}{4}y = x^2 - \frac{3}{4}x - \frac{5}{4},$

or $-\frac{3}{4}y = x^2 - \frac{3}{4}x + \frac{9}{64} - \frac{5}{4} - \frac{9}{64},$

or $-\frac{3}{4}y = (x - \frac{3}{8})^2 - \frac{89}{64},$

or $y = -\frac{4}{3}(x - \frac{3}{8})^2 + \frac{89}{48}.$

$p = -\frac{4}{3}; \qquad q = \frac{3}{8}; \qquad r = \frac{89}{48}.$

The coördinates of the highest point are $\frac{3}{8}$ and $\frac{89}{48}$.

(*b*) $y = x^2 + 4x - 6.$

(*c*) $3y = x^2 + 6x + 10.$

(*d*) $3y = 4x^2 + 6x + 10.$

(*e*) $4x^2 = y - 2x - 6.$

(*f*) $x^2 + y = 6.$

(*g*) $x^2 + 2x + y + 1 = 0.$

(*h*) $2x^2 - 5x - 6y + 10 = 0.$

(*i*) $5x^2 + 2x - y = 0.$

31. The Parabola. All curves drawn and discussed in the preceding section have equations reducible to the form $y = p(x - q)^2 + r$. They have a line of symmetry, $x = q$, and change from an increasing to a decreasing curve or *vice versa* at the point (q, r).[2] Each curve is the curve $y = px^2$ translated horizontally a distance q, and vertically a distance r.[3] Any curve which is of the shape of $y = px^2$ is called a **parabola.** The line of symmetry of the curve is called the **axis** of the parabola, and the point of intersection of the curve with its axis is called the **vertex.**[4]

[1] A line such that line segments drawn perpendicular to it and terminated by the curve, are bisected by it is called a **line of symmetry of the curve.** Thus any line passing through the center of a circle is a line of symmetry for the circle.

A point such that line segments drawn through it and terminated by the curve, are bisected by it, is called a **point of symmetry.** Thus, the center of a circle is a point of symmetry.

[2] A curve is said to be **increasing** when its ordinate increases as its abscissa increases; and **decreasing** when its ordinate decreases as its abscissa increases. The increase in each case being understood to be the algebraic increase.

[3] Distance here means algebraic distance. When q is positive the curve is moved to the right, when negative to the left. When r is positive the curve is moved up, when negative down.

[4] It can be shown that the parabola is the locus of a point moving in a plane so that its distance from a fixed line, **the directrix,** is always equal to its distance from a fixed point, **the focus.** The parabola is also the curve of intersection of a right cir-

Any equation of the form $y = \alpha x^2 + \beta x + \gamma$, where α, β and γ are constants, positive or negative, represents a parabola with its axis parallel to the Y-axis; for

$$\begin{aligned}\alpha x^2 + \beta x + \gamma &= \alpha\left(x^2 + \frac{\beta}{\alpha}x\right) + \gamma \\ &= \alpha\left(x^2 + \frac{\beta}{\alpha}x + \frac{\beta^2}{4\alpha^2}\right) + \gamma - \frac{\beta^2}{4\alpha} \\ &= \alpha\left(x + \frac{\beta}{2\alpha}\right)^2 + \frac{4\alpha\gamma - \beta^2}{4\alpha},\end{aligned}$$

which is of the form $p(x - q)^2 + r$, where $p = \alpha$, $q = \dfrac{-\beta}{2\alpha}$, and $r = \dfrac{4\alpha\gamma - \beta^2}{4\alpha}$. The expresson $\alpha x^2 + \beta x + \gamma$ is the most general expression of the second degree in x, or it is a general quadratic. Thus we see that the equation $y = \alpha x^2 + \beta x + \gamma$ represents a parabola with axis parallel to the Y-axis, and by completing the square in the x terms we are able to locate its vertex. Problems illustrating this point have already been given in exercise 22, § **30.**

We have already seen that the slope of the tangent line drawn to the curve $y = px^2$ is $2px$, *i.e.*, $2p$ times the x-coördinate of the point of contact of the tangent line. Another way of saying this is that the y is changing $2px$ times as fast as the x. From this and what was said in the preceding paragraph we see that the slope of the tangent line drawn to the curve for $y = \alpha x^2 + \beta x + \gamma$ is $2p(x - q)$, or $2p$ times the distance of the point of contact of the tangent line from the axis of the parabola. Conversely, it may be shown that *if a number y changes $2p(x - q)$ times as fast as the number x, x and y are connected by the parabolic law.*

Exercises

1. Plot data given in Table VIII. Plot temperature along the horizontal axis and weight along the vertical axis.

cular cone and a plane parallel to an element of the cone. The parabola has many useful and interesting properties, one of which is: If a parabola be rotated about its axis forming a surface of revolution, the **paraboloid of revolution**, rays of light passing out from its focus are reflected by the surface into lines parallel to the axis of the paraboloid. This property is made use of in the construction of reflectors.

2. An empirical equation of the second degree in T for the data given in Table VIII is $W' = 0.06T^2 - 0.36T + 0.47$. Here W' is used in order to distinguish between weight computed by formula and observed weight, W.

Copy Table VIII upon a sheet of paper, form $M7$. In a third column headed W' place weights computed by the empirical formula. In a third column headed $W - W'$ place the differences between the observed and computed weights. Plot W' upon the drawing of exercise 1. Mark the points representing W' so that they are readily distinguished from the points representing W.

TABLE VIII.—RELATION BETWEEN TEMPERATURE AND WEIGHT OF 10 LITERS OF WATER

T = temperature in degrees centigrade.

W = loss in weight, measured in grams, of 10 liters of water, as temperature differs from 4°.

T	W	T	W
0	1.3	16	10.3
2	0.3	18	13.8
4	0.0	20	17.7
6	0.3	22	22.0
8	1.2	24	26.8
10	2.7	26	31.9
12	4.8	28	37.1
14	7.3	30	43.3

3. Plot data given in Table IX.[1] Plot $\frac{ld}{D}$ along the horizontal axis using 1 cm. to represent one-tenth unit. Plot velocities along the vertical axis, using 1 cm. to represent two one-hundredths of 1 foot per second. Sketch a smooth curve among the points. The curve may not pass through all the plotted points.[2] Begin to number the vertical axis with 2.90. The true origin will be far below the sheet of paper.

[1] The velocity at any point of a moving stream is determined by a current meter placed at that point.

[2] This curve is called a vertical velocity curve. In practical work, however, velocities are plotted along the horizontal axis and depths along the vertical axis, and down from the origin. Your drawing gives the usual form of plotting if it is turned 90° in a clockwise direction. Vertical velocity curves are parabolic in shape, with the axis of the parabola parallel to the surface of the water.

TABLE IX.—RELATION BETWEEN VELOCITY AND DEPTH AT A POINT IN THE LOWER MISSISSIPPI

Depth at observed point = d; whole depth = D.

$\frac{d}{D}$	Velocity, feet per second	$\frac{d}{D}$	Velocity, feet per second
0.0	3.195	0.5	3.228
0.1	3.230	0.6	3.181
0.2	3.253	0.7	3.127
0.3	3.261	0.8	3.059
0.4	3.252	0.9	2.976

4. An empirical formula connecting velocity, V, and depth, x, is $V' = -0.78(x - 0.296)^2 + 3.261$. Compute V' from this formula and tabulate your results following, in a way, directions given in exercise 2.

5. A strip of tin L feet long and 40 inches wide is made into a trough with rectangular cross section by bending up an equal portion of each side. Find the width of each portion bent up such that the volume of the trough is a maximum.

HINT: The volume is a maximum when the area of the cross section is a maximum. Let y be the area of the cross section. Let x be the width of the portion bent up. The dimensions of the cross section are then x and $40 - 2x$. Hence

$$y = x(40 - 2x)$$
$$y = -2(x^2 - 20x)$$
$$y = -2(x^2 - 20x + 100) + 200$$
$$y = -2(x - 10)^2 + 200$$

The equation is that of a parabola with its high point at (10, 200). Thus the maximum area, 200, occurs when 10 inches of tin are turned up.

6. Same as exercise 5, but with the width 60 inches and length equal to 10 feet. Find the volume of the trough.

32. The Equilateral Hyperbola.

Exercise

1. Plot upon squared paper a curve for

$$(a)\ y = \frac{1}{x} \qquad (b)\ y = \frac{2}{x} \qquad (c)\ y = \frac{3}{x}.$$

Choose the origin near the center of the sheet and plot for negative as well as for positive values of x. Let 1 cm. (or 1/2 inch) represent the unit along each axis. When x is numerically small the interval between successive values assigned to it must be small in order to obtain points near enough together for sketching the curve. Compare curves (*a*), (*b*), and (*c*).

These curves are equilateral hyperbolas. They are special cases of the more general equilateral hyperbola, $y = \frac{a}{x}$, where a is a constant. When one quantity, y, varies with another, x, so that the relation connecting the two is represented by the equation $y = \frac{a}{x}$, **y is said to vary inversely with x.**

If points plotted upon squared paper arrange themselves along a curve which seems to be an equilateral hyperbola, $y = \frac{a}{x}$ may, as a first assumption, be taken as an empirical equation connecting the variable numbers plotted. The problem now becomes one of selecting a value for the constant a, and of determining the agreement between the values of the numbers plotted and those called for by the equation. If the a cannot be selected giving a fairly close agreement between observed and calculated numbers, it means that the law we are dealing with in nature is not that of one number varying inversely as the other.

An easy elementary method for determining the a, as well as showing whether or not the law is that of the inverse power, will be illustrated by means of an example.

In Table X are given the relative volumes of a mass of air subjected to pressures ranging from 0 to 100 pounds per square inch measured above atmospheric pressure. The temperature of the air is constant, 60° Fahrenheit.

Plot the data upon a sheet of squared paper. Plot pressure along the Y-axis, using 2 cm. (or 1 inch) to represent 10 pounds. Plot volume along the X-axis, using 10 cm. (or 5 inches) to represent the unit volume. This curve seems to be of the general shape of $y = \frac{a}{x}$ excepting that it crosses or touches the X-axis at unity instead of remaining above the X-axis and becoming nearer and nearer to it.

To find an equation connecting x (volume) and y (pressure) proceed as follows. Construct a third column of numbers for Table X, by taking reciprocals of the numbers representing relative volumes given in the second column. Call these values the x' values. (The prime is put upon the x to distinguish it from the x which represents the true volume reading.) Plot a curve be-

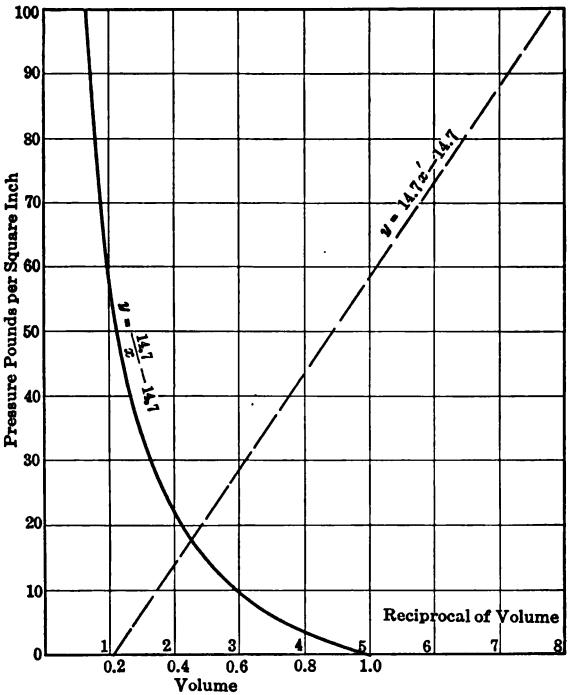


FIG. 22.

tween pressure and x'. Let 2 cm. (or 1 inch) along the Y-axis represent 10 pounds; let 2 cm. (or 1 inch) along the horizontal axis represent one unit of x'. Such a curve, reduced, is given in Fig. 22. This curve is a straight line with slope equal to 14.7 and y-intercept equal to -14.7. The equation of the

line is $y = 14.7x' - 14.7$. Since $x' = \frac{1}{x}$, the relation connecting volume and pressure is represented by the equation $y = \frac{14.7}{x} - 14.7$. It is readily seen that this is the equation of the curve $y = \frac{14.7}{x}$ lowered 14.7 units; that the first curve plotted between pressure and volume is an equilateral hyperbola; and that the pressure readings given in Table X, when increased by 14.7, vary inversely as the volume.

TABLE X.—AIR COMPRESSION FROM ONE ATMOSPHERE AT SEA-LEVEL

Pressure, lb. per sq. in.	Volume	Pressure, lb. per sq. in.	Volume
0	1,000	45	0.246
1	0.936	50	0.227
2	0.880	55	0.210
3	0.830	60	0.196
4	0.786	65	0.184
5	0.746	70	0.173
10	0.595	75	0.163
15	0.495	80	0.155
20	0.423	85	0.147
25	0.370	90	0.140
30	0.328	95	0.134
35	0.295	100	0.128
40	0.268	.....	.....

The pressure readings given in the table were taken by means of a pressure gauge which registers the difference in pressure between the outside and inside of the receptacle containing the air. If we assume Boyle's law for the compressibility of a gas (the volume varies inversely as the pressure) the y-intercept, 14.7, found above, shows that when the experiment was conducted the pressure gauge was registering 14.7 pounds less than the true pressure exerted upon the air, or that the atmospheric pressure was 14.7 pounds per square inch.

If, in Fig. 22, the plotted points had not arranged themselves along a straight line the method outlined in § **28** could have been

used in finding the slope and intercept. When the plotted points arrange themselves along a curve which is not a straight line, it may mean that the original variables are not connected by the equilateral hyperbolic law.

Exercise

Find the equation connecting I and W, using the data given in Table XI,[1] where I represents the indicated horse power of a steam engine, and W the number of pounds of steam consumed per hour.

TABLE XI

I	W	I	W
36.8	460.0	15.8	222.8
21.5	406.2	12.6	182.7
26.3	344.5	8.4	137.0
21.0	279.3	.....	.....

[1] Taken from Elementary Practical Mathematics by John Perry.

CHAPTER II

LOGARITHMS

33. Logarithm Defined. The equation $y = x^2 - 1$ is solved for y. If values are assigned to x, the corresponding values of y may be calculated. The same equation may be written $x = \pm\sqrt{y+1}$. Here the equation is solved for x, and if values are assigned to y, the corresponding values of x may be calculated. The two equations represent the same law connecting x and y.

Exercises

1. Solve each of the following equations for x:

(*a*) $y = x$;
(*b*) $y = x - 1$;
(*c*) $y = 2x + 3$;
(*d*) $y = \frac{1}{2}x - 6$;
(*e*) $y = x^2$;
(*f*) $y = 2x^2$;
(*g*) $y = x^2 - 2x + 1$;
(*h*) $y = 3x^2 + 6x + 3$;
(*i*) $y = 3x^2 - 2x + 1$;
(*j*) $y = x^3$;
(*k*) $y = x^3 - 1$;
(*l*) $y = 3x^3 - 3$;
(*m*) $y = x^3 + 3x^2 + 3x + 1$;
(*n*) $y = x^3 - 3x^2 + 3x$;
(*o*) $y = \pm\sqrt{x^2 - 25}$.

2. Solve each of the following equations for x and also for y:

(*a*) $x^2 + y^2 = 25$;
(*b*) $x^2 - y^2 = 25$;
(*c*) $x^2 + 2x + y^2 = 25$;
(*d*) $x^2 - 2x + y^2 = 25$;
(*e*) $x^2 + y^2 - 2y = 25$;
(*f*) $x^2 + 2x + y^2 - 2y = 25$;
(*g*) $x^2 + x + y^2 - 2y = 16$;
(*h*) $xy = 1$;
(*i*) $xy = 2$;
(*j*) $xy + y = 1$;
(*k*) $xy^2 = 3$;
(*l*) $x^2y = 5$;
(*m*) $x^2y + xy^2 = 3$;
(*n*) $x^2y + xy + xy^2 = 0$;
(*o*) $xy^3 = 1$;
(*p*) $x^3y^3 = -1$.

All equations of the preceding exercise can be solved for x by means of the symbols and operations of elementary algebra. The equation $y = a^x$, where a is a known positive constant different

from zero, cannot, however, be solved for x by means of these elementary operations. The best we can do is to say, x = the exponent of the power to which a must be raised to produce y. In this verbal sense, the equation is now solved for x. The words "exponent of the power to which a must be raised to produce y" are abbreviated by **"logarithm of y to the base a,"** and in writing they are further abbreviated by **"$\log_a y$."** Thus, the equation $y = a^x$, solved for x, gives $x = \log_a y$. Log_a is then a new symbol; considered as an operator, operating upon y, it gives the number, x, such that when a has x for an exponent, the power equals y. a is called the **base.** The definition then of the logarithm of a number x to the base a is: **"the exponent of the power to which a must be raised to produce the number in question."**

Of the two equivalent equations, $y = a^x$ and $x = \log_a y$, the first is called the exponential form, and the second the logarithmic form.

Exercises

1. Write the following equations in logarithmic form:

(*a*) $100 = 10^2$;	(*e*) $25 = 5^2$;	(*i*) $1 = 10^0$;
(*b*) $10 = 10^1$;	(*f*) $9 = (\sqrt{3})^4$;	(*j*) $2 = (\sqrt{2})^2$;
(*c*) $4 = 2^2$;	(*g*) $16 = 2^4$;	(*k*) $3 = (\sqrt{3})^2$;
(*d*) $8 = 2^3$;	(*h*) $125 = 5^3$;	(*l*) $8 = (\sqrt{2})^6$;
(*m*) $2^0 = 1$;		(*n*) $5^0 = 1$.

2. Write the following equations in exponential form:

(*a*) $\log_2 4 = 2$;	(*f*) $\log_{10} 10 = 1$;	(*k*) $\log_3 9 = 2$;
(*b*) $\log_{10} 100 = 2$;	(*g*) $\log_3 3 = 1$;	(*l*) $\log_{10} 1000 = 3$;
(*c*) $\log_{10} 1 = 0$;	(*h*) $\log_5 5 = 1$;	(*m*) $\log_{10} 10{,}000 = 4$;
(*d*) $\log_2 1 = 0$;	(*i*) $\log_5 25 = 2$;	(*n*) $\log_5 1 = 0$;
(*e*) $\log_3 1 = 0$;	(*j*) $\log_2 32 = 5$;	(*o*) $\log_{99} 1 = 0$.

3. Give the value of each of the following:

(*a*) $\log_2 4$;	(*h*) $\log_{10} 100$;	(*o*) $\log_{10} 0.1$;
(*b*) $\log_3 9$;	(*i*) $\log_{10} 1000$;	(*p*) $\log_{10} 0.01$;
(*c*) $\log_3 27$;	(*j*) $\log_{10} 10{,}000$;	(*q*) $\log_{10} 0.001$;
(*d*) $\log_5 25$;	(*k*) $\log_{10} 1$;	(*r*) $\log_{10} 100{,}000{,}000$;
(*e*) $\log_4 4$;	(*l*) $\log_3 1$;	(*s*) $\log_{10} 0.000000001$.
(*f*) $\log_6 6$;	(*m*) $\log_5 1$;	
(*g*) $\log_{10} 10$;	(*n*) $\log_a 1$;	

4. Evaluate the following:

(*a*) $\log_2 2 + \log_4 4 + \log_5 5$;
(*b*) $\log_3 9 + \log_5 25 + \log_5 125 + \log_6 36$;
(*c*) $\log_{10} 1000 + 3\log_6 36 - 5\log_2 16 + 2\log_5 625$;
(*d*) $\log_3 1 - \log_3 1 + \log_3 3 - \log_{10} 10 + \log_3 27$;
(*e*) $\frac{1}{2}\log_{10} 100 - \frac{2}{3}\log_{10} 10 - \frac{1}{2}\log_{10} 0.1 - 2\log_{10} 0.01$;
(*f*) $\log_a a$; (*g*) $\log_a 1$.

34. Theorem: The logarithm of the product of two numbers is the sum of their separate logarithms.

Proof: Let

$$\log_a u = x \qquad \text{and} \qquad \log_a v = y$$

Then

$$a^x = u \qquad \text{and} \qquad a^y = v$$

By multiplication

$$a^{x+y} = uv,$$

or

$$\log_a uv = x + y,$$

or

$$\log_a uv = \log_a u + \log_a v$$

Thus the theorem: *The logarithm of the product of two numbers is the sum of their separate logarithms.*

It is easily seen that *the logarithm of the product of three or more numbers is the sum of their separate logarithms.*

If no base is indicated, the base 10 *is understood.*

Exercises

If $\log 2 = 0.3010$, $\log 3 = 0.4771$, $\log 5 = 0.6990$, and $\log 7 = 0.8451$, find the logarithms of the following:[1]

(*a*) 6	(*g*) 35	(*m*) 9
(*b*) 15	(*h*) 40	(*n*) 27
(*c*) 10	(*i*) 4	(*o*) 18
(*d*) 20	(*j*) 8	(*p*) 24
(*e*) 45	(*k*) 16	(*q*) 30
(*f*) 21	(*l*) 32	(*r*) 12.

[1] The logarithms used in this book are taken from a four-place logarithmic table and are correct to the nearest fourth decimal place.

35. Theorem: The logarithm of the quotient of two numbers is the logarithm of the dividend diminished by the logarithm of the divisor.

Proof: Let $\log_a u = x$ and $\log_a v = y$.
Then

$$a^x = u \text{ and } a^y = v.$$

By division

$$a^{x-y} = \frac{u}{v},$$

or

$$\log_a\left(\frac{u}{v}\right) = x - y,$$

or

$$\log_a\left(\frac{u}{v}\right) = \log_a u - \log_a v.$$

Thus the theroem: *The logarithm of a quotient is equal to the logarithm of the dividend diminished by the logarithm of the divisor.*

Exercises

Using the logarithms given in preceding Exercise, find the logarithms of the following: (*a*) $\frac{3}{2}$; (*b*) $\frac{7}{3}$; (*c*) $\frac{4}{5}$; (*d*) $\frac{7}{5}$; (*e*) $\frac{7}{2}$; (*f*) $\frac{5}{2}$.

36. Theorem: The logarithm of the power of a number is the logarithm of the number multiplied by the index of the power.

Proof: Let $\log_a u = x$.
Then

$$u = a^x,$$

Then

$$u^n = a^{nx},$$

or

$$\log_a u^n = nx,$$

or

$$\log_a u^n = n \log_a u.$$

Thus the theorem: *The logarithm of the power of a number is equal to the logarithm of the number multiplied by the index of the power.*

The theorem is true for fractional exponents as well as for integral exponents, and from it then follows the law that *the logarithm of the*

root of a number is equal to the logarithm of the number divided by the index of the root.

ILLUSTRATION:

If $\log 2 = 0.3010$, $\log 4 = 2 \log 2 = 2 \times 0.3010 = 0.6020$, and $\log \sqrt{2} = \frac{1}{2} \log 2 = \frac{1}{2} \times 0.3010 = 0.1505$.

Exercises

1. Using the logarithms of the number given for the exercises, **§ 34,** find the logarithms of the following: (*a*) 3^2; (*b*) 3^3; (*c*) 5^2; (*d*) 27; (*e*) 125; (*f*) $\sqrt{5}$; (*g*) $3^{\frac{3}{2}}$; (*h*) $7^{\frac{1}{4}}$; (*i*) $125^{\frac{1}{2}}$; (*j*) 625; (*k*) $\sqrt[3]{35}$; (*l*) $(\frac{3}{16})^{\frac{1}{2}}$; (*m*) $(\frac{21}{35}^{0})^{\frac{1}{8}}$.

2. If the log 2 is 0.3010, what is the (*a*) log 20? (*b*) log 200? (*c*) log 2000? (*d*) log 20,000? (*e*) log 200,000? How do these logarithms differ? What part of these logarithms is the same? Why?

3. If log 3 = 0.4771, what is the (*a*) log 30? (*b*) log 300? (*c*) log 3000? (*d*) log 30,000? What part of these logarithms is the same? Why?

4. If log 373 = 2.5717, what is the log 37.3? log 3.73? log 3730? log 37,300? What part of these logarithms is the same? Why? If a number is greater than unity, how does moving the decimal point affect its logarithm?

37. Characteristic and Mantissa. If 10 is used as base, the following equations are true:

$$\begin{aligned}
\log\ 10{,}000 &= 4 \\
\log\ 1{,}000 &= 3 \\
\log\ 100 &= 2 \\
\log\ 10 &= 1 \\
\log\ 1 &= 0 \\
\log\ 0.1 &= \log 1 - \log 10 = -1 = 9 - 10 \\
\log\ 0.01 &= \log 1 - \log 100 = -2 = 8 - 10 \\
\log\ 0.001 &= \log 1 - \log 1{,}000 = -3 = 7 - 10 \\
\log\ 0.0001 &= \log 1 - \log 10{,}000 = -4 = 6 - 10
\end{aligned}$$

From the above table, which can be extended indefinitely in both directions, we see that any number greater than 1000 and less than 10,000 has a logarithm lying between 3 and 4, *i.e.*, the logarithm is 3 plus some (decimal) fraction. All numbers between

1000 and 10,000 have four digits preceding the decimal point, and all numbers, excepting 1000, having four digits preceding the decimal point lie between 1000 and 10,000. Therefore, the logarithm of any number, excepting 1000, having four digits preceding the decimal point is 3 plus a (decimal)fraction.

In a similar way it may be shown that the logarithm of any number which is not an integral power of 10, consists of an integral part plus a positive decimal fraction. For fractional numbers the integral part of the logarithm is usually written in the form of a binomial, as $9 - 10$, $8 - 10$, $7 - 10$, etc.

The integral part of a logarithm is called the **characteristic.** The positive decimal part is called the **mantissa.** The mantissa of a positive integral power of 10 or of the reciprocal of a positive integral power of 10 is zero.

If the log 373 = 2.5717:

$$\begin{aligned}
\log 3.73 &= \log 373 - \log 10 = 2.5717 - 1 = 1.5717 \\
\log 3.73 &= \log 37.3 - \log 10 = 1.5717 - 1 = 0.5717 \\
\log 0.373 &= \log 3.73 - \log 10 = 0.5717 - 1 = 9.5717 - 10 \\
\log 0.0373 &= \log 0.373 - \log 10 = 9.5717 - 10 - 1 = 8.5717 - 10 \\
\log 0.00373 &= \log 0.0373 - \log 10 = 8.5717 - 10 - 1 = 7.5717 - 10
\end{aligned}$$

The logarithms of numbers between 1 and 0 are negative. It will be noticed from the above illustration of the method of writing negative logarithms that *the mantissa is always the same for the same sequence of digits, and is entirely independent of the position of the decimal point.*

The characteristic of the logarithm of a number depends only upon the position of the decimal point in the number, and is entirely independent of the sequence of the significant digits. A rule for determining the characteristic of the logarithm of a number is: *the characteristic of the logarithm of a number, using 10 as a base, equals the number of places the first significant figure is removed from the units' place, and is positive if the first significant figure is to the left of the units' place, and is negative if it is to the right of units' place.*

Thus the characteristic of the logarithm of 37.3 is $+ 1$; of 3.73 is 0; of 0.373 is $- 1$, or $9 - 10$; of 0.0373 is $- 2$, or $8 - 10$, etc.

Exercises

1. Give the characteristic of the logarithm of each the following numbers:

(*a*) 375; (*e*) 87.36; (*h*) 0.6127; (*k*) 17.31;
(*b*) 172.6; (*f*) 8.276; (*i*) 0.08217; (*l*) 0.00082;
(*c*) 37.62; (*g*) 0.561; (*j*) 0.00756; (*m*) 0.000002.
(*d*) 186.2;

2. Locate the decimal point in the numbers for which the following are logarithms:

(*a*) 3.7283; (*d*) 1.8725; (*g*) 3.1786; (*j*) 7.6862 − 10;
(*b*) 3.6872; (*e*) 0.8271; (*h*) 8.1268 − 10; (*k*) 9.3621 − 10.
(*c*) 2.7826; (*f*) 9.1267 − 10; (*i*) 0.3675;

3. Do you see any reason for using 10 as a base rather than any other number?[1]

[1] When 10 is used as base the system of logarithms is called the **common system.** The system of logarithms employed in theoretical investigation has for its base the sum of the infinite series

$$2 + \frac{1}{2} + \frac{1}{2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \cdots$$

This number is represented by the letter e, and to seven decimal places is 2.7182818. When the base e is used the logarithm is called **the natural, the hyperbolic,** or **the Naperian** logarithm. The number e enters into a great many formulas representing laws of nature. To illustrate: if the number of times faster y is changing than x is proportional to y, it can be shown that the law connecting y and x is $y = ce^{kx}$ where c and k are constants. This law is sometimes called the compound interest law. The *rate of growth of bacteria* is proportional to the number of bacteria. The formula giving the number at any time is then the compound interest formula.

It is an easy matter to find the logarithm of a number to the base e, if we know its logarithm to the base 10. For, let $\log_e N = x$, where N is the number whose natural logarithm is desired. Then

$$e^x = N$$

$$e = N^{\frac{1}{x}}.$$

Taking the common logarithm, we obtain:

$$\log e = \frac{1}{x} \log N,$$

solving for x

$$x = \frac{\log N}{\log e},$$

whence

$$\log_e N = \frac{\log N}{\log e}.$$

The $\log e = 0.434294$. Then $\log_e N = 2.302585 \log N$. This formula converts the common logarithm of the number N into its natural logarithm. To find the common logarithm of N if its natural logarithm is given we use the formula, $\log N = 0.434294 \log_e N$.

4. If the log 2 = 0.3010, log 3 = 0.4771, log 7 = 0.8451, and log 11 = 1.0414, find the logarithms of the following:

(*a*) $\frac{121}{15}$. (*b*) $\frac{70}{144}$. (*c*) $\frac{84\sqrt{24}}{55}$.

(*d*) $\frac{0.35 \cdot 121^2}{21 \cdot 1.25}$. (*e*) $\left(\frac{2.5 \cdot 350}{16}\right)^{\frac{1}{4}}$. (*f*) $\frac{0.07}{0.064}$. (*g*) $\left(\frac{500}{6}\right)^{\frac{1}{3}}$.

5. Fill in the blanks of the following table:

Number	Logarithm	Base	Number	Logarithm	Base
100	2	.....	32	.....	$\sqrt{2}$
125	.....	5	81	5.0	.....
81	4	.....	16	2.0	.....
.....	5	2	17	5.0	.....
.....	27	3	$\sqrt{3}$	.....	$\sqrt{2}$
.....	5	10	$3\sqrt{3}$	1.5	.....
625	.....	5	.....	.....	.....

38. Logarithmic Tables. A table of common logarithms consists of the mantissas of the logarithms of numbers put in tabular form. Since the characteristic of the common logarithm of a number is dependent only upon the position of the decimal point, it is not published in the tables, but is supplied by the computer; and since the mantissa is independent of the position of the decimal point, no decimal points are placed in the numbers in the table. Thus, if one were to find the logarithm of 27.6 from a table of common logarithms, he would enter the table with 276, take out the corresponding mantissa, which is a decimal fraction, and prefix the characteristic 1.

The tables commonly used are known as four-place, five-place, and six-place; meaning by this that the mantissas are given, respectively, to four, five, and six decimal places. In a four-place table the number interval is usually 1 in the third place; in a five-place table, 1 in the fourth place; and in a six-place

table, 1 in the fifth place. If one wishes to compute accurately to four, five, or six places, he should use, respectively, four-, five-, or six-place tables. Do not use a five- or six-place table if a four-place table will suffice to give accurately the required number of places.

The use of logarithms and logarithmic tables is primarily to shorten the work of numerical computation, and to relieve the computer of as much mental work as possible.

In the explanations of the use of logarithmic tables, it is assumed that the student is using Slichter's four-place tables.[1] All numerical work, unless otherwise stated, is to be done accurately to four figures.

39. To Find the Logarithm of a Number from the Tables. On the first page of the tables will be found the logarithms (*i.e.*, the mantissas) of numbers from 1 to 1000. There are nine sets of vertical columns. The first column of each set is headed **No.** and contains numbers whose logarithms are given directly opposite in the second column, headed **log.** In the third column, headed **d,** are found the differences (called **tabular differences**) of the consecutive mantissas given in the second column. From the table one finds the mantissa for the sequence of digits 376 to be 0.5752. Hence

$$\begin{aligned}
\log\ 3760 &= 3.5752 \\
\log\ 376 &= 2.5752 \\
\log\ 37.6 &= 1.5752 \\
\log\ 3.76 &= 0.5752 \\
\log\ 0.376 &= 9.5752 - 10 \\
\log\ 0.0376 &= 8.5752 - 10 \\
\log\ 0.00376 &= 7.5752 - 10
\end{aligned}$$

Exercises

Find the logarithm of each of the following numbers:

1. (*a*) 365. (*b*) 36.5. (*c*) 3.65. (*d*) 3650.

2. (*a*) 725. (*b*) 72.6. (*c*) 7.26. (*d*) 7260.

[1] "Slichter's logarithmic and trigonometric tables" are printed both in pamphlet form and upon four leaves of cardboard bound together along the longer edge. For desk use the cardboard form should be used.

If a student does any considerable amount of numerical work he cannot afford to use poorly arranged and slow tables, such as are frequently found in text-books on trigonometry and algebra.

3.	(*a*) 816.	(*b*) 81.6.	(*c*) 8.16.	(*d*) 8160.
4.	(*a*) 261.	(*b*) 26.1.	(*c*) 2.61.	(*d*) 2610.
5.	(*a*) 113.	(*b*) 1.13.	(*c*) 0.113.	(*d*) 0.0113.
6.	(*a*) 216.	(*b*) 0.216.	(*c*) 0.00216.	(*d*) 216,000.
7.	(*a*) 101.	(*b*) 1010.	(*c*) 0.0101.	(*d*) 0.000101.
8.	(*a*) 2.	(*b*) 0.2.	(*c*) 0.02.	(*d*) 0.00002.
9.	(*a*) 9.99.	(*b*) 99.9.	(*c*) 0.999.	(*d*) 9,990,000.
10.	(*a*) 0.0136.	(*b*) 0.000136.	(*c*) 136,000.	(*d*) 136,000,000.

In writing a decimal fraction without an integral part before the decimal point, *always* place a zero before the decimal point; as, 0.1276; 0.0281; 0.1201; 0.0026. In writing and reading the mantissas of logarithms do not omit the last one or more digits, even though they be zero. Thus, write:

log 399 as 2.6010 *not* 2.601
log 414 as 2.6170 *not* 2.617
log 455 as 2.6580 *not* 2.658

In reading the mantissas of the logarithms of numbers, read two digits at a time, *not* one at a time. Thus, read 2.7162, "two, (pause) seventy-one, sixty-two;" *not*, "two, point, seven, one, six, two."

If the number consists of more than three digits the method of finding the mantissa of its logarithm from the table is illustrated by the following examples. Find the log 376.8. From the table

$$\log 376 = 2.5752$$
$$\log 377 = 2.5763$$

The difference between these logarithms is 11 (disregarding the decimal point). This difference is given in the table in the column headed **d.** We see that a difference of 1 in the number produces a difference of 11 in the logarithm. A difference of 0.8 in the number will then produce (approximately) a difference of 11×0.8, or 8.8 (use 9) in the logarithm. Then, since the log 376 = 2.5752, the log 376.8 = 2.5752 + 0.0009 = 2.5761. The 9 added to the logarithm of 376 is called the **correction** for 8.

Find log 521.7.

$$\log 521 = 2.7168$$

Correction: 9×0.7 $= 6$

$\log 521.7 = 2.7174$

Find log 8618.

$\log 8610 = 3.9350$

Correction for 8 $= 4$

$\log 8618 = 3.9354$

Exercises

Find the logarithms of the following numbers:

1.	(*a*) 172.6.	(*b*) 17.26.	(*c*) 1.726.	(*d*) 17,260.
2.	(*a*) 276.5.	(*b*) 27.65.	(*c*) 2.765.	(*d*) 27,650.
3.	(*a*) 365.8.	(*b*) 3.658.	(*c*) 0.3658.	(*d*) 3,658,000.
4.	(*a*) 812.6.	(*b*) 0.8126.	(*c*) 0.08126	(*d*) 812,600.
5.	(*a*) 8.887.	(*b*) 8887.	(*c*) 0.008887.	(*d*) 0.8887.

If there are five digits in the number the correction may be found by multiplying the tabular difference by the last two digits of the number. But, with the exception of some of the numbers occurring near the beginning of the table where the tabular differences are large, no error will be introduced in the logarithm if the number is fist reduced to four significant figures. Thus, to four decimal places, the log 86326 is the same as log 86330.

40. Tables of Proportional Parts. In finding corrections the tabular difference is multiplied by the digit in the fourth place of the number, or by the last two digits of a five-place number. This multiplication is performed by means of the tables of proportional parts, the tables headed **p. p.** Thus, to find the correction for the log 37,267, turn to the **p. p.** table headed 16, 16 being the tabular difference. Opposite 6 we find 9.6, the correction for 6 in the fourth place; opposite 7 we find 11.2, the correction for 7 in the fourth place. But 7 is in the fifth place, therefore the correction for it is 1.12. Add mentally 9.6 and 1.12, and obtain for the complete correction 10.72 (use 11).

In taking out the mantissa from the tables, the addition of corrections *should be performed mentally*. A good method to follow is illustrated by example. To find log 13.78. First write down the characteristic, 1. Then, with the table at your left, find 137 in the **No.** column and mark the corresponding mantissa by placing the thumb nail above, or the finger nail below it.

Do not read this mantissa, but read the tabular difference, 32. From the **p. p.** tables find the correction, 26, for 8. Now, go back to the mantissa marked by the finger nail, and read it increased by 26, *i.e.*, 1393. Then place 1393 after the characteristic 1 previously written down.

41. Arrangement of Work. All work should be arranged in a vertical column and done with pen and ink. Make the digits about 1/8 inch tall and space them horizontally about eight to the inch. Study the formula in which you are substituting and decide upon some arrangement of your work in the vertical column which will make the additions, subtractions, etc., of logarithms as systematic and easy as possible. Fill out the vertical column as far as possible before turning to the table of logarithms. This is called **blocking out the work.** The method of blocking out the work is illustrated by the following example: Find the area, in acres, of a triangular piece of land, the sides a, b and c being, respectively, 127.6, 183.7, and 201.3 rods. The formula giving the number of acres, A, is[1]

$$A = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{160}$$

[1] The proof of this formula, as generally given, presupposes a knowledge of trigonometry. A proof involving only algebra and geometry is given here.

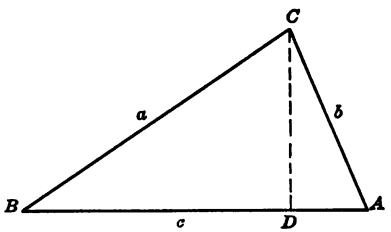


FIG. 23.

Let ABC, Fig. 23, be the triangle whose sides are a, b, and c. Let CD be drawn perpendicular to AB. Then

$$CD^2 = a^2 - (c - AD)^2 \quad (1)$$

and

$$AD = \sqrt{b^2 - CD^2} \quad (2)$$

Substituting in (1),

$$CD^2 = a^2 - c^2 + 2c\sqrt{b^2 - CD^2} - b^2 + CD^2, \quad (3)$$

where s is one-half the sum of the three sides. We may block out the work as follows:

$$\begin{aligned} s &= 256.3 \\ a &= 127.6 \\ b &= 183.7 \\ c &= 201.3 \\ 2s &= 512.6 \\ s - a &= 128.7 \\ s - b &= 72.6 \\ s - c &= 55.0 \\ \log s &= 2.4087 \\ \log (s - a) &= 2.1096 \\ \log (s - b) &= 1.8609 \\ \log (s - c) &= 1.7404 \\ \text{sum} &= 8.1196 \\ 1/2 \text{ sum} &= 4.0598 \\ \log 160 &= 2.2041 \\ \log A &= 1.8557 \\ A &= 71.73 \end{aligned}$$

In performing the work the expressions on the left of the equality signs were all written down first. Then the numerical

---

or

$$a^2 - c^2 - b^2 = -2c\sqrt{b^2 - CD^2}. \tag{4}$$

Squaring (4),

$$(a^2 - c^2 - b^2)^2 = 4c^2(b^2 - CD^2),$$

or

$$4c^2b^2 - (a^2 - c^2 - b^2)^2 = 4c^2\,CD^2.$$

Factoring,

$$(2cb + a^2 - c^2 - b^2)(2cb - a^2 + b^2 + c^2) = 4c^2\,CD^2,$$

or

$$[a^2 - (c - b)^2][(c + b)^2 - a^2] = 4c^2\,CD^2.$$

Factoring,

$$(a + b - c)(a - b + c)(c + b + a)(c + b - a) = 4c^2\,CD^2,$$

or

$$(2s - 2c)(2s - 2b)(2s)(2s - 2a) = 4c^2\,CD^2,$$

or

$$4s(s - a)(s - b)(s - c) = c^2\,CD^2,$$

or

$$\frac{cCD}{2} = \sqrt{s(s - a)(s - b)(s - c)}$$

= area of triangle.

values of a, b, and c were filled in and $2s$, s, $s - a$, $s - b$, and $s - c$ computed. Then the logarithms of s, $s - a$, $s - b$, $s - c$, and 160 were taken from the tables. Then the logarithms of s, $s - a$, $s - b$, $s - c$ were added, giving what is called the "sum." Then one-half of this was taken giving what is called "1/2 sum," and from it the logarithm of 160 was subtracted giving log A. From the tables of anti-logarithms[1] the number 71.73 was taken out corresponding to the mantissa 8557, the characteristic 1 fixing the decimal point after the second digit.

Exercises

Compute the areas, in acres, of the following triangles. a, b, and c are expressed in rods. Record the time spent upon each exercise.

1. $a = 93.6$
$b = 101.7$
$c = 127.3.$

2. $a = 216.7$
$b = 172.5$
$c = 216.3.$

3. $a = 103.7$
$b = 156.4$
$c = 217.7.$

4. $a = 101.73$
$b = 127.27$
$c = 131.38.$

[5.] Plot a logarithmic curve. Plot numbers along the X-axis and logarithms along the Y-axis. Let x range from 0.1 to 10. Let 2 cm. (or 1 inch) on the X-axis and 10 cm. (or 5 inches) on the Y-axis represent one unit. Is this curve concave on the upward or on the downward side? If the correction of a logarithm falls on a 0.5 shall we call it another one or drop the 0.5? Why? What would be a similar rule in taking out the number from its logarithm? Why?

[6.] With the aid of the T-square, triangle, compasses, and scale, draw to scale a triangle with the sides given in exercise 1. Let 1/2 inch represent 10 rods. To do this use the scale marked 20 on the triangular boxwood scale. Drop a perpendicular from any vertex upon the opposite side. Measure the length of this perpendicular. Multiply its length (expressed as rods) by the length of the base (expressed as rods) and divide by 320. How could you get a more accurate determination of the area?

[7.] Same as exercise 6, but use data given in exercise 2.

[8.] Same as exercise 6, but use data given in exercise 3.

[1] A table of anti-logarithms is given on the last page of the Slichter table. From it the number corresponding to a given mantissa is obtained. If the use of this table is not apparent, the student should read the directions given on the first cover.

[9.] Same as exercise 6, but use data given in exercise 4.

[10.] Upon a sheet of paper of good quality make a drawing according to the following directions. Exercise great care in making this drawing.

Draw a horizontal line 10 inches long. Divide it into ten equal parts by drawing vertical lines 1/4 inch long *on the lower side only*. Subdivide each of these ten intervals into ten equal parts by drawing vertical lines 1/8 inch long on the lower side of the horizontal line. We now have a unit of length (10 inches) divided into 100 equal parts. In other words, the distance between any two consecutive vertical lines represents one one-hundredth of one unit. Mark the longer vertical lines beginning at the left-hand end 0.0, 0.1, 0.2, 0.3, . . ., 0.9, 1.0. Upon the upper side of the horizontal line and at each end draw a vertical line 1/4 inch long. Mark the one at the left 1, and the one at the right 10. Since the logarithm of 1 is 0, and of 10 is 1, it will be seen that the lines on the upper side of the horizontal line representing the numbers 1 and 10 stand opposite their logarithms. The logarithm of 2 is 0.3010. Find the location of this number (estimating the position for the third place) on the lower scale, and on the upper side draw a vertical line 1/4 inch long. Mark this line 2. In a similar way draw lines to be marked 3, 4, 5, . . ., 8, and 9, standing opposite the points of the lower scale representing their logarithms. In a similar manner draw vertical lines 1/8 inch long representing every tenth of a unit between 1 and 2, 2 and 3, and so on. We now have a double scale. The lower is uniform, and the upper, called a **logarithmic scale,** is non-uniform. Make a second drawing by placing the uniform scale above and the non-uniform scale, the logarithmic scale, below.

11. Find the values of the following, using logarithms:

(*a*) $(27.83)^{\frac{1}{3}}(1.621)^{5}$.
(*b*) $\sqrt{(72.18)^{3}}\,\sqrt{1.382}$.
(*c*) $(0.6382)^{2} \div 1.782$.
(*d*) $\sqrt{1.278} \div 0.0183$.
(*e*) $(0.003782)^{\frac{1}{3}}(1.7821)^{\frac{1}{3}}$.
(*f*) $0.0007826 \div 0.007936$.
(*g*) $[3.782 \times 26.78 \div 37.92]^{\frac{1}{3}}$.
(*h*) $[3.728 \times 17.83 \div (2.675)^{2}]^{\frac{1}{3}}$.
(*i*) $\sqrt{1.186}\,(0.02763)^{2} \div (0.001726)^{3}$.

12. Find the amount of \$10.00 in ten years at compound interest at 6 percent; (*a*) interest compounded annually; (*b*) interest compounded semi-annually.

42. Further Suggestions for Arrangement of Logarithmic Work. If in a single formula several substitutions are to be made, similar operations should be performed for each substitution before

another is begun. This will be illustrated by the following exercises:

1. Table XII indicates, in part, the amount of milk delivered at the University of Wisconsin Creamery from Feb. 16, 1911, to March 15, 1911, inclusive.

TABLE XII

Pat. No.	Milk, lb.	Ave. test	Price, cents	Pat. No.	Milk, lb.	Ave. test	Price, cents
4	4075	3.6	27	38	8228	3.9	27
5	749	3.5	27	39	2434	3.85	27
2	2395	3.8	27	63	6822	4.00	27
3	4046	3.6	27	80	1784	3.45	26
6	1503	3.6	26	65	1628	3.7	27
7	2020	4.15	27	102	1038	3.7	27
8	9012	3.6	26	105	705	4.15	27
9	5750	3.4	26	111	2646	3.9	27
10	1491	3.8	26	86	97	4.8	27
83	738	4.05	27	44	1603	4.6	27
36	2060	3.6	27	81	2291	3.65	27
37	2258	3.8	27	82	777	4.1	27

From these data the amount due each patron is to be computed by multiplying each hundredweight of milk delivered, by the average test and then by the price. For the numerical work use a sheet of computation paper, form *M7*. In the first column, the column marked *O*, on the first, second, third, fourth, fifth, and sixth lines write, respectively, "pat. No.," "log of lb.," "log of test," "log of price," "log of amount," and "amount." Then on the first line of the remaining columns put in from left to right the patron's number, as 4, 5, 2, 3, etc. Place one number in each vertical column. If there are not enough vertical columns fasten two or more sheets together. On the next line fill in the logarithms of the number of hundredweights of milk delivered by the patrons. On the next line fill in the logarithms of the numbers representing the average test of the patron's milk. On the line below this fill in the logarithms of the price paid each patron per pound of butter fat. On the next line fill in the sums of the three logarithms above. Then take out and place on the next line the anti-logarithms of the

sums, and the numerical work is complete. By this arrangement all numerical work pertaining to each patron is placed in a vertical column headed by his number. The order of the work should be as outlined above, so that all the operations of any one kind are completed before those of another kind are begun. As an illustration, the work for patron No. 65 will appear as follows:

65
1.2117
0.5682
9.4314 − 10
1.2113
16.27

2. A solid of revolution[1] is 10 feet long and has flat ends normal to its axis of revolution. Its circumferences at various distances from its larger end were measured and the results are given in Table XIII. The numbers in the first column represent the dis-

TABLE XIII

Distance, feet	Circumference, feet	Distance, feet	Circumference, feet
0	10.00	5	6.28
1	9.00	7	5.47
2	8.15	8	5.00
3	7.40	9	4.68
4	6.80	10	4.40

tances, in feet, from the larger end to the points at which the measurements were taken. The measurements are given in the second column. Compute the area of the corresponding eleven cross sections, arranging the numerical work upon a sheet of paper, form *M*7, as follows: On the first horizontal line in the first column of the sheet write "distance;" on the second line write "log cir.;" on the next line write "2 log cir.;" on the next line write "log area;" and on the line below this write "area." In the columns headed I, II, III, etc., fill in from left to right, on the first horizontal line, the distances, as 0, 1, 2, . . . , 9, and 10. On the

[1] A solid of revolution is the volume bounded by a surface which may be generated by rotating a curve 360° about a straight line.

line below this place the logarithms of the circumferences. Multiply each of these logarithms by 2 and place the product in the next horizontal line. Write the logarithm of $\frac{1}{4\pi}$ on the lower edge of a slip of paper. (The logarithm of $\frac{1}{4\pi}$ is to be computed separately, the work placed in systematic form on the lower portion of the calculation sheet.) Place the slip of paper so that the logarithm of $\frac{1}{4\pi}$ stands directly over the value of 2 log cir. in column I. Add logarithm of $\frac{1}{4\pi}$ and the value of 2 log cir., placing the sum on the fifth line. Slide the slip of paper to the second column and add as before. Slide the paper to the third column and add, then to the next, and so on for the eleven columns. Take out the anti-logarithms corresponding to the logarithms on the fifth line, and the numerical work is complete. Plot a curve connecting area and distance. Plot distance along the X-axis, using 2 cm. (or 1 inch) to represent 1 foot. Plot square feet along the Y-axis, using 2 cm. (or 1 inch) to represent 1 square foot. Find the area under this curve by counting the squares. This area will be proportional to the volume of the solid. Why? What is the volume expressed in cubic feet?

3. $BCB'D$, Fig. 47, represents the intersection of a right circular cylinder with a plane perpendicular to the elements of the cylinder; and $BAB'A'$ the intersection with a plane inclined 30° to the first plane. The two planes intersect in the line BB' cutting the axis of the cylinder at the point O. Our problem is to construct the curve $BPAB'$. Take any point R upon the line $B'OB$. Let $OR = x$. Draw RQ and RP, respectively in the normal and inclined planes, perpendicular to BOB'. P and Q, the points where the two perpendiculars pierce the cylinder, are upon the same element of the cylinder. PRQ is a 60° right triangle with the angle R equal to 30°. Represent RP by y', and RQ by y. Then, from the property of the 60° triangle,

$$\frac{y'}{y} = \frac{2}{\sqrt{3}},$$

or

$$y' = 1.155y.$$

But, since $BQCB'$ is a circle, $y = \sqrt{a^2 - x^2}$, where a is the radius of the cylinder. Then

$$y' = 1.155\sqrt{a^2 - x^2},$$

or

$$y' = 1.155\sqrt{(a - x)(a + x)}.$$

By giving to x, (OR), a series of values ranging from 0 to the length of the radius, a, the corresponding perpendicular distances RP, (y'), may be computed and the curve $BPAB'$ plotted.

Let a equal 2 feet. Using the above formula, compute y' for the values $x = 0$, $x = 0.1$, $x = 0.2$, . . . , $x = 1.8$, $x = 1.9$, and $x = 2$. Plot a curve between x and y', *i.e.*, the curve $BPAB'$.[1]

4. A barn is provided with two 3½-foot diameter cupola ventilators connected with the basement. The intake ventilators in the concrete basement wall are 7 inches in diameter, made by using 7-inch stovepipe and 7-inch elbows as forms in the construction of the wall. Neglecting friction, and assuming no air leaks in around the doors and windows, develop a formula giving n, the required number of intakes, so that the velocity of the air in the intakes shall be the same as the velocity in the flues of the cupolas. Compute n without using logarithms.

43. To find the Anti-logarithm from the Table of Logarithms. The use of the anti-logarithmic table is for instructional purposes only. Since the method of taking out, from an anti-logarithmic table, the number corresponding to a given logarithm is the same as that for finding the logarithm if the number is given, the anti-logarithmic table enables the student to become familiar with the one method before he is to learn how to use a logarithmic table backward, so to speak, in finding anti-logarithms.

An anti-logarithmic table has its advantages, but these are more than offset by advantages gained by using the logarithmic table backward. Five- and six-place tables are rarely, if ever, provided with anti-logarithmic tables.

[1] The curve thus plotted if drawn full sized would be a pattern (not allowing for the lap in seam) for the saddle of a 4-foot circular ventilator, if the rafters of the barn make an angle of 60° with the vertical.

Later an easier method will be given for drawing this curve as well as a method for constructing a pattern for cutting the sheet iron for the flue of the ventilator, and for the conical roof.

The method of taking out the number from the logarithmic table which corresponds to a given logarithm will be illustrated by example.

Find x if $\log x = 2.1786$. By taking the mantissa, 1786, to the table of logarithms, we see that the number x is between 150 and 151 (disregarding decimal point). Thus we can write down at once the first three digits, 150, of the number and whatever correction is to be made are digits suffixed. The given mantissa is 25 greater than the mantissa for 150, and the tabular difference is 29. Hence the required number must be 25/29 of one unit greater than 150. Since $25/29 = 0.86$, the number corresponding to the mantissa 1786 is 15086, or $x = 150.86$. The decimal point is placed after the third digit to correspond to the characteristic 2.

The division of 25 by 29 may be effected by using the proportional parts table. In the proportional part table headed 29 the nearest we can get to 25 on the smaller side is 23.2, which corresponds to 8. This means that in dividing 25 by 29 the first digit in the quotient is 0.8, with a remainder of $25 - 23.2 = 1.8$. Again in the **p.p.** table we have 17.4 opposite 6, which means that the nearest second digit of the quotient of 25 by 29 is 6. Thus, at once from the proportional parts table we read that 25 divided by 29 is 0.86.

From the above illustration the student is not to infer that with four-place logarithms one can compute accurately to five places. This is, in general, not true. In general, only four significant figures of the number should be found. If the number lies near the beginning of the table where the tabular differences are large, the fifth digit may have significance and should then be calculated. In this connection the student should read section 72.

Exercises

Find, from the table of logarithms, the numbers corresponding to the following logarithms. In finding corrections make use of the **p.p.** tables.

1. 2.7864.	**2.** 3.1286.	**3.** 1.8152.
4. 9.6278 − 10.	**5.** 8.1278 − 10.	**6.** 6.1785 − 10.

CHAPTER III

THE CIRCULAR FUNCTIONS: THE TRIANGLE

44. Angular Magnitude. If a straight line, as AB, Fig. 24, rotates about a fixed point O until it occupies some other (or the same) position, as $A'B'$, an angular magnitude is generated, namely, the amount of rotation. All angles are or may be imagined to have been generated in this way. The initial position of the line is called the **initial side,** and the terminal position the **terminal side** of the angle. The point of intersection of the terminal and the initial sides is called the **vertex** of the angle. This concept of angular magnitude is more general than that of elementary geometry in that the angular magnitude here defined is not only unlimited in size, but has a *sense; i.e.*, the angular magnitude represents either counter-clockwise or clockwise rotation. If the rotation is counter-clockwise the angle or the rotation is called **positive;** if clockwise, **negative.**

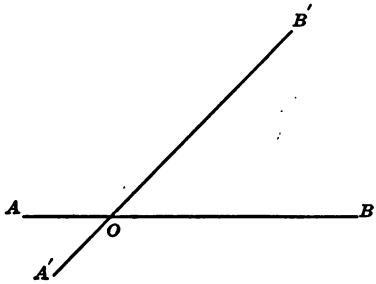


FIG. 24.

If we fix our attention upon a particular spoke of a rotating fly wheel and note its position at the beginning and at the end of a definite interval of time, we say that this spoke has rotated through an angle, the magnitude and sense of which is the amount and direction of the rotation noted. If we had fixed our attention upon a point of the wheel and had imagined it connected to the center of rotation by means of a straight line, this imaginary line would have rotated through the same angle as the spoke. We say that the point has rotated through an angle. Since every point of the

wheel rotates through the same angle in any given interval of time, we say the wheel has rotated through an angle.

The following is another illustration of a body rotating through an angle. Suppose the telescope of a transit is first pointed at one object and then at a second; we say that the vertical planes through the two lines of sight form an angle with each other. This same thought is expressed by saying that the line of sight of the transit has been rotated through an angle. The magnitude and sense of this angle is the amount and direction of rotation.

The **practical unit** for measuring angular magnitude is the one three hundred and sixtieth part of one complete revolution. This unit is called the **degree.** The degree is divided into sixty equal parts called **minutes.** The minute is divided into sixty equal parts called **seconds.**

Since the magnitude and sense of an angle is unchanged when it is placed in different positions of its plane, we may move any given angle until its vertex coincides with the origin of the system of rectangular coördinates, and until its initial side coincides with the positive part of the X-axis.

If the angle is positive and less than 90° its terminal side will fall within the first quadrant, and the angle is said to be **in the first quadrant.** If the angle is positive and greater than 90°, but less than 180°, its terminal side will fall within the second quadrant and the angle is said to be **in the second quadrant.** If the angle is positive and greater than 180°, but less than 270°, its terminal side will fall within the third quadrant and the angle is said to be **in the third quadrant.** Similarly, a positive angle greater than 270° but less than 360° is **in the fourth quadrant;** if greater than 360° but less than 450°, **in the first quadrant;** and so on. If the angle is negative and numerically less than 90°, it is **in the fourth quadrant;** if numerically greater than 90°, but less than 180°, **in the third quadrant;** and so on. In general, an angle is said to be in that quadrant within which its terminal side would fall, if its vertex were placed at the origin, and if its initial side were placed on the positive part of the axis of x.

Thus, in Fig. 25, a, b, and c, represent angles in the first quadrant; d, e, and f, angles in the second quadrant; g, h, and i, angles in the fourth quadrant; and j, k, and l, angles in the third

quadrant. In each case the arrow indicates the sense and magnitude of the rotation. a, b, d, g, h, i, and k are positive angles.

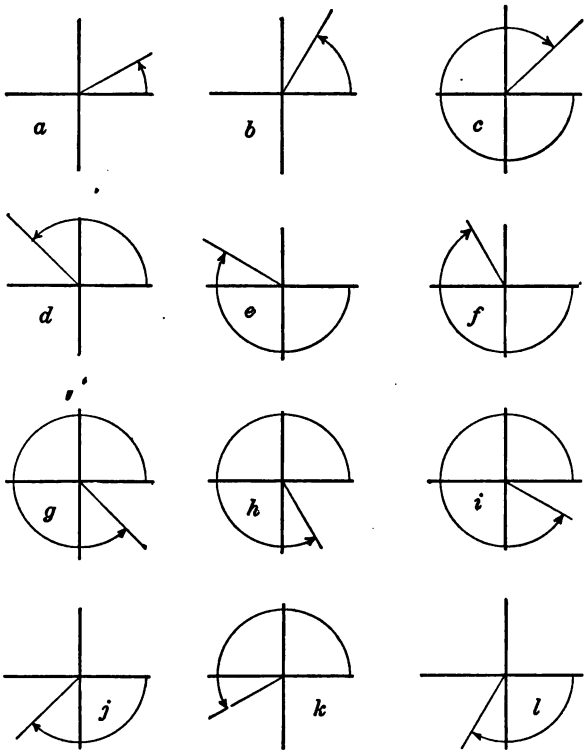


FIG. 25.

Exercises

1. In what quadrants are the following angles?

(a) 16°; (d) 217°; (g) 87°; (j) − 76°; (m) − 276°;
(b) 291°; (e) 796°; (h) 102°; (k) − 376°; (n) − 5°.
(c) 368°; (f) 560°; (i) 212°; (l) − 175°;

2. With the aid of the T-square and the 60° and the 45° triangles,

draw lines through the origin, making positive angles of 15°, 30° 45°, 60°, 75°, 90°, 105°, 120°, 135°, 150°, 165°, and 180° with the positive direction of the axis of x.

3. In a 60° triangle what is the ratio (a) of the length of the hypotenuse to the length of the shorter leg? (b) of the length of the hypotenuse to the length of the longer leg? (c) of the length of the shorter leg to the length of the longer leg? (d) of the length of the longer leg to the length of the hypotenuse?

4. In a 45° triangle, what is the ratio of the length of the hypotenuse to the length of a leg?

5. Draw an equilateral triangle 6 inches on a side. From each vertex draw a line to the middle of the opposite side. Mark the numerical size of all the angles formed by these lines.

6. Draw a right triangle, one acute angle of which is 60°, and the hypotenuse of which is 6 inches in length. What is the length of the shorter leg of this triangle? What is the length of the longer leg? What is the ratio of the length of the hypotenuse to the length of the shorter leg? to the length of the longer leg? What is the ratio of the length of the longer leg to the length of the shorter leg?

8. Same as exercise 7 but draw the hypotenuse 10 inches long.

9. What answers of exercises 7 and 8 are the same? Will these answers remain the same for any length of hypotenuse? Why?

10. Draw a right triangle with hypotenuse equal to 6 inches in length, and an acute angle equal to 45°. What is the length of the legs? What is the ratio of the length of the hypotenuse to the length of a leg? What is the ratio of the length of one leg to the length of the other leg?

11. Same as exercise 10 but make the length of the hypotenuse 10 inches.

12. What answers in exercises 10 and 11 are the same? What answers will remain the same for any length of the hypotenuse? Why?

13. Draw a right triangle, one acute angle of which is 15°, and the hypotenuse 6 inches in length. What is the length of the shorter leg? of the longer leg? What is the ratio of the length of the hypotenuse to the length of the shorter leg? of the length of the hypotenuse to the length of the longer leg? of the length of the longer leg to the length of the shorter leg?

14. Same as exercise 13, but make the hypotenuse 10 inches long.

15. What answers are the same in exercises 13 and 14? What answers will remain the same for any length of the hypotenuse? Why?

45. Trigonometric Functions Defined. Construct any angle in the first quadrant. (This angle should be general, and not one previously constructed in the preceding exercises.) Choose any point, P, upon the terminal side, and from it drop a perpendicular upon the initial side; *i.e.*, upon the X-axis. We now have a right triangle with the vertex of an acute angle (the original angle constructed) at the origin. This triangle is called the **triangle of reference** for the constructed angle. The three sides, a, b, and c, of this triangle may give rise to six ratios, viz.:

$$\frac{a}{c}, \frac{b}{c}, \frac{a}{b}, \frac{b}{a}, \frac{c}{b}, \text{ and } \frac{c}{a}.$$

From a property of similar triangles it is seen that these ratios are entirely independent of the size of the triangle; *i.e.*, independent of the position of the point P upon the terminal side, provided, however, that the angle at the origin does not change. If, however, the angle is made to increase or decrease by any amount, no matter how small, each of the six ratios will change in value.[1] Since these ratios depend for their values upon the size of the angle and are entirely independent of the size of the triangle of reference, they are functions[2] of the angle, and are called the **trigonometric, or circular, functions.**

In the triangle of reference the side opposite the given angle (the angle whose vertex is at the origin) is called the opposite (*opp.*) side. The side adjacent the angle is called the adjacent (*adj.*) side.

The six trigonometric ratios are defined as follows. α is the angle in question:

The ratio $\frac{opp.}{hyp.}$ is called sine α; abbreviated by sin α.

[1] The increase or decrease in the size of the angle should not be so large that the angle will not remain in the first quadrant. Later on it will be seen that the above statement may not be true if the terminal side of the angle were to move to another quadrant.

[2] One quantity, y, is said to be a function of another quantity x, if, when x is known, y is determined. Thus, the area of a circle is a function of its radius; for if the radius is known, the area is determined. In this case, since we know the formula connecting the radius and the area of a circle, we can actually compute the area. If an angle in the first quadrant is known, the six ratios among the lengths of the three sides of the triangle of reference are determined.

The ratio $\frac{adj.}{hyp.}$ is called cosine α; abbreviated by cos α.

The ratio $\frac{opp.}{adj.}$ is called tangent α; abbreviated by tan α.

The ratio $\frac{adj.}{opp.}$, or the reciprocal of tan α, is called cotangent α; abbreviated by cot α.

The ratio $\frac{hyp.}{adj.}$, or the reciprocal of cos α, is called secant α; abbreviated by sec α.

The ratio $\frac{hyp.}{opp.}$, or the reciprocal of sin α, is called cosecant α; abbreviated by csc α.

It is to be remembered that in the order given above, the first and last, the second and fifth, and the third and fourth ratios are reciprocals.

The side opposite and the side adjacent are, respectively, the y-coördinate and the x-coördinate of P, the point of the terminal side from which the perpendicular was let fall upon the X-axis. Since the point P is in the first quadrant, so that its x- and y-coördinates are both positive, the side opposite and the side adjacent of the triangle of reference are considered positive. The length of the hypotenuse is the distance of the point P from the origin, and is always considered positive.

If the angle is in the second quadrant, a triangle of reference is formed as above by dropping a perpendicular from any point P of the terminal side upon the X-axis (the initial side produced through the vertex). The six ratios of the lengths of the sides of this triangle are called the trigonometric functions of the angle and are named the same as if the angle were in the first quadrant. In this case, however, the side adjacent, the x-coördinate of the point P, is negative; hence the cosine, the tangent, the cotangent, and the secant are negative.

In a similar manner, if the angle is in the third quadrant, we see that both the adjacent and opposite sides of the triangle of reference are negative, and the sine, the cosine, the secant, and the

cosecant are negative. If the angle is in the fourth quadrant, the sine, the tangent, the cotangent, and the cosecant are negative.

Exercises

1. In what quadrant or quadrants is the angle if:

(*a*) The sine negative? (*b*) The tangent positive?
(*c*) The secant negative? (*d*) The cosine positive?
(*e*) The cosecant positive?
(*f*) The sine negative and the tangent positive?
(*g*) The secant positive? (*h*) The sine positive?
(*i*) The tangent positive and the cosine negative?
(*j*) The cosine negative and the sine positive?
(*k*) The cosine negative?
(*l*) The sine negative and the cosecant negative?
(*m*) The cosine negative and the cosecant negative?
(*n*) The cosine positive and the tangent negative?
(*o*) The secant negative and the cotangent positive?
(*p*) The cotangent positive and the tangent positive?
(*q*) The sine negative and the tangent positive?
(*r*) The cotangent positive and the cosecant negative?
(*s*) The secant positive and the cosecant negative?
(*t*) The cosecant positive and the tangent positive.
(*u*) The cosine negative and the tangent positive?
(*v*) The cotangent negative and the secant positive?
(*w*) The sine negative and the secant positive?
(*x*) The sine positive and the cosecant negative?
(*y*) The tangent negative and the cotangent positive?
(*z*) The cosine negative and the secant positive?

2. Give the values of the six trigonometric functions of the following angles. Tabulate your results. Do *not* reduce radicals to approximate decimal fractions. (*a*) 30°; (*b*) 45°; (*c*) 60°; (*d*) 120°; (*e*) 135°; (*f*) 150°; (*g*) 210°; (*h*) 225°; (*i*) 240°; (*j*) 300°; (*k*) 315°; (*l*) 330°.

3. At 117 feet from the base of a flagstaff standing on level ground, the angle of elevation[1] of the top of the staff is 45°; what is

[1] If a point *A* is at a higher elevation than a point *B*, the **angle of elevation** of the point *A* at *B*, is the angle between the horizontal and a line drawn from *B* to *A*. If a point *A* is at a lower elevation than a point *B*, the **angle of depression** of the point *A* at *B*, is the angle between the horizontal and a line drawn from *B* to *A*.

the height of the staff? What is the height if the angle of elevation is 60°? If 30°?

4. In exercise 15, page 11 how high above the plates is the point of meeting of the upper edges of the upper and the lower rafters? How far from the outside of the building will a plumb-line dropped from this point strike the floor? How high above the plates is the ridge of the barn?

5. The diagonal of a rectangular plot of ground is 82 rods. One side is 41 rods. What is the length of the other side?

6. The sides of a rectangular plot of ground are 100 and 173.2 rods. What is the approximate length of the diagonal?

7. The sine $\alpha = \frac{3}{5}$; the cosine is positive. Find the values of the remaining five functions.

HINT: Since the sine and cosine are both positive the angle is in the first quadrant, and all functions are positive. A triangle whose hypotenuse is five units in length and whose opposite side is three units in length (hence the adjacent side four units in length) is a triangle of reference for α. Hence we have, $\cos \alpha = \frac{4}{5}$, $\tan \alpha = \frac{3}{4}$, $\cot \alpha = \frac{4}{3}$, $\sec \alpha = \frac{5}{4}$, and $\csc \alpha = \frac{5}{3}$.

8. Find the values of the trigonometric functions, construct to scale a triangle of reference, and measure the angle with a protractor, for each of the following:

(*a*) $\sin \alpha = \frac{3}{7}$, and the cosine is negative.
(*b*) $\cos \alpha = \frac{2}{7}$, and the tangent is negative.
(*c*) $\tan \alpha = \frac{1}{2}$, and the sine is negative.
(*d*) $\sin \alpha = -1\frac{5}{8}$, and the tangent is negative.
(*e*) $\sec \alpha = 2$, and the tangent is negative.
(*f*) $\sin \alpha = \frac{3}{2}$, and the cosine is negative.
(*g*) $\cos \alpha = \frac{3}{2}$, and the sine is negative.
(*h*) $\sec \alpha = \frac{2}{3}$, and the sine is negative.

9. At two points, A and B, 100 feet apart and in line with the base of a tower standing on level ground, the angles of elevation of the top are 45° and 30°, respectively. Find the height of the tower.

10. In exercise 9, if the angles of elevation are 60° and 30°, what is the height of the tower, and how far are the points A and B from its base?

46. Functions of 0°, 90°, 180°, and 270°. The functions of 0°, 90°, 180° and 270° are not included in the above definition of trigonometric functions, since for these angles there are no triangles of reference. We shall define the functions of 0°, 90°, 180°,

and 270°, as the limits toward which the functions of any angle, say α, approach as the angle α approaches 0°, 90°, 180°, or 270°. Thus, if the angle α is in the first quadrant, and if the hypotenuse of the triangle of reference is kept constant as the angle becomes smaller and smaller, the opposite side will become smaller and smaller and approach zero. The sine of this variable angle will then approach zero as the angle approaches zero, or the $\sin 0° = 0$. The length of the adjacent side approaches the length of the hypotenuse as the angle approaches zero, or the limit of the cosine is 1, or $\cos 0° = 1$. As the angle approaches zero, the cotangent and the cosecant increase without limit.[1]

By considering α a small negative angle (hence in the fourth quadrant) the cotangent and cosecant are both negative and increase numerically without limit as α approaches zero. These facts together with what was said above regarding cotangent and cosecant are expressed by writing $\cot 0° = \pm \infty$ and $\csc 0° = \pm \infty$.

Thus we have:

$$\begin{array}{lll} \sin 0° = 0, & \tan 0° = 0, & \sec 0° = 1, \\ \cos 0° = 1, & \cot 0° = \pm \infty & \csc 0° = \pm \infty. \end{array}$$

By allowing the angle to approach 90°, the student will show that:

$$\begin{array}{lll} \sin 90° = 1, & \tan 90° = \pm \infty & \sec 90° = \pm \infty, \\ \cos 90° = 0, & \cot 90° = 0, & \csc 90° = 1. \end{array}$$

[1] If b is a variable which approaches zero as its limit, and if a is a constant or a variable whic approaches a limit c, the fraction $\frac{a}{b}$ increases without limit. Thus, if $a = 1$, and if b approaches zero by taking successively the values 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, . . ., the fraction has successively the values 1, 2, 3, 4, . . . By giving to b a value sufficiently small, the value of the fraction, $\frac{a}{b}$, may be made larger, and for subsequent values of b remains larger, than any assigned number. We then say that $\frac{a}{b}$ increases without limit as b approaches zero.

The expression, "r increases without limit," or "increases indefinitely," as b approaches zero is usually written

$$r = \infty$$

The student is not to infer that ∞ (infinity) is a number or the limit of a number. The expression, $r = \infty$, is a symbol representing, "r increases without limit," or "r increases indefinitely."

The student will further show that:

$\sin 180° = 0$, $\tan 180° = 0$, $\sec 180° = -1$,
$\cos 180° = -1$, $\cot 180° = \pm\infty$, $\csc 180° = \pm\infty$,

and that:

$\sin 270° = -1$, $\tan 270° = \pm\infty$, $\sec 270° = \pm\infty$
$\cos 270° = 0$, $\cot 270° = 0$, $\csc 270° = -1$

Exercises

1. Tabulate the functions of 0°, 90°, 180° and 270°.

2. Find the values of the functions of 360°; of 450°; of 540°; of 630°; of 720°; of 810°.

3. Is it possible for an angle to have a sine equal to 1/2? to 3/4? to -2?

4. Is it possible for an angle to have a cosine equal to 1/2? to 3/4? to -2?

5. Is it possible for an angle to have a tangent equal to 1/2? to 3/4? to -2?

6. Is it possible for an angle to have a cotangent equal to 1/2? to 3/4? to -2?

7. Is it possible for an angle to have a secant equal to 1/2? to 3/4? to -2?

8. Is it possible for an angle to have a cosecant equal to 1/2? to 3/4? to -2?

9. As the angle varies from 0° to 360° in a positive sense, how do the values of the sine, the cosine, the tangent, the cotangent, the secant and the cosecant change?

10. In the equation of the straight line, $y = ax + b$, a is what trigonometric function of what angle?

11. Compare the $\sin 30°$ with $\sin(-30°)$; $\sin 45°$ with $\sin(-45°)$; $\sin 60°$ with $\sin(-60°)$; $\sin 135°$ with $\sin(-135°)$; $\sin 300°$ with $\sin(-300°)$; $\sin\alpha$ with $\sin(-\alpha)$.

12. Compare $\cos 30°$ with $\cos(-30°)$; $\cos 60°$ with $\cos(-60°)$; $\cos 210°$ with $\cos(-210°)$; $\cos\alpha$ with $\cos(-\alpha)$.

13. Compare $\tan 45°$ with $\tan(-45°)$; $\tan 150°$ with $\tan(-150°)$; $\tan 330°$ with $\tan(-330°)$; $\tan\alpha$ with $\tan(-\alpha)$.

14. Compare $\cos 30°$ with $\sin 60°$; $\sin 30°$ with $\cos 60°$; $\tan 30°$ with $\cot 60°$; $\sec 30°$ with $\csc 60°$; $\sin 45°$ with $\cos 45°$; $\tan 45°$ with $\cot 45°$; $\cos\alpha$ with $\cos(90° - \alpha)$; $\sin\alpha$ with $\cos(90° - \alpha)$; $\tan\alpha$ with $\cot(90° - \alpha)$; $\sec\alpha$ with $\csc(90° - \alpha)$; $\csc\alpha$ with $\sec(90° - \alpha)$, where α is an angle in the first quadrant.

15. Compare sin 135° with sin 45°; sin 120° with sin 60°; cos 120° with cos 60°; tan 120° with tan 60°; sin 150° with sin 30°; cos 150° with cos 30°; tan 150° with tan 30°; sin 240° with sin 60°; cos 240° with cos 60°; tan 240° with tan 60°; sin 300° with sin 60°; cos 300° with cos 60°; tan 300° with tan 60°; sin $(180° - \alpha)$ with sin α; cos $(180° - \alpha)$ with cos α; tan $(180° - \alpha)$ with tan α; sin $(180° + \alpha)$ with sin α; cos $(180° + \alpha)$ with cos α; tan $(180° + \alpha)$ with tan α; sin $(360° - \alpha)$ with sin α; cos $(360° - \alpha)$ with cos α; tan $(360° - \alpha)$ with tan α.

16. Express the following as functions of some positive angle less than 90°: sin 112°; cos 127°; tan 171°; sin 92°; cos 170°; tan 131°; cot 175°.

HINT: Construct an angle of 112°. Its terminal side falls in the second quadrant. Draw a triangle of reference. Construct an angle of 22°. Draw a triangle of reference for 22°. These two triangles of reference are similar, and their homologous sides are in proportion. Whence: sin 112° = + cos 22°; cos 112° = − sin 22° (because cosine is positive in the first quadrant, and negative in the second), and tan 112° = − cot 22°.

17. Show that

$$\tan \alpha = \frac{\sin \alpha}{\cos \alpha}$$

and that

$$\cot \alpha = \frac{\cos \alpha}{\sin \alpha}.$$

47. Even and Odd Functions. Let us consider the function x^2 (*i.e.*, $y = x^2$). If we give to x any two values numerically equal, one positive and the other negative, the two corresponding values of x^2 (*i.e.*, of y) are equal. The curve for x^2 (*i.e.*, $y = x^2$), plotted upon squared paper is symmetrical with respect to the Y-axis. The same can be said of the functions x^4, x^6, x^8, $x^2 + 3x^4$, and $x^2 + 5x^4 - 6x^6 - 7$. These functions are said to be **even.** An **even function** is one whose value remains unchanged when the sign of the argument is changed.

Let us next consider the function x^3 (*i.e.*, $y = x^3$). If we give to x two values numerically equal but with opposite signs, the two corresponding values of x^3 (*i.e.*, of y) are numerically equal but of opposite signs. The curve for x^3 (*i.e.*, $y = x^3$), plotted upon squared paper is symmetrical with respect to the origin. The

same can be said of the functions x^5, x^7, x, $x + 3x^3 - 7x^5$, and $x^3 - 2x^5 + 6x^7$. These functions are said to be **odd.** An **odd function** is one whose value remains numerically the same but changes sign when the sign of the argument is changed.

In previous exercises the fact was developed that **cos** α and **sec** α are **even functions,** and that **sin** α, **csc** α, **tan** α and **cot** α are **odd functions.** These facts are shown graphically in **§ 49.**

48. Circular Measure. The circular unit of angular magnitude is defined as an angle whose sides intercept an arc equal in length to the radius of a circle when the vertex of the angle is placed at the center of the circle. This unit is called the **radian.**[1] This system of angular measure is fundamental in mechanics, physics, and pure mathematics. It must be thoroughly mastered by the student.

Since the circumference of a circle is $2\pi r$, where r is the radius of the circle, there are 2π radians in one complete revolution; or

$$360° = 2\pi \text{ radians}$$
$$180° = \pi \text{ radians}$$
$$90° = \frac{\pi}{2} \text{ radians, etc.}$$

To find the number of radians in an angle expressed in degrees, divide the number of degrees by 180 (this gives the number of straight angles in the given angle) and multiply the quotient by π, the number of radians in a straight angle. Thus, to change 176° to radians, we have $\frac{176}{180}\pi = 3.07$; or $176° = 3.07$ radians.

To find the number of degrees in an angle expressed in radians, divide the number of radians by π (this gives the number of straight angles in the given angle) and multiply this quotient by 180 (the number of degrees in a straight angle). Thus, to change 4.6 radians to degrees we have $\frac{(4.6)\ (180)}{\pi} = 263.6$, or 4.6 radians equals 263.6°.

[1] If θ is the angle measured in radians at the center of a circle of radius r, the length of the arc intercepted by the angle is $r\theta$, and the area of the circular sector is $\frac{1}{2}r^2\theta$.

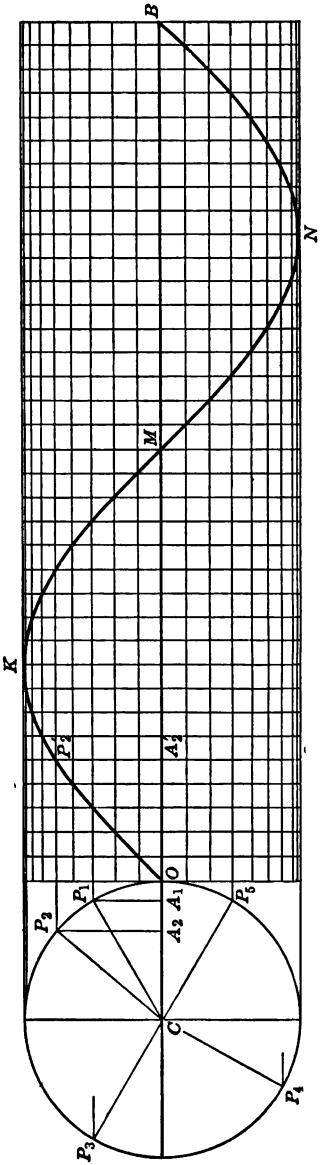


FIG. 26.—The sine curve.

Exercises

1. Express in radians the following: (*a*) 135°; (*b*) 45°; (*c*) 161°; (*d*) 275°.

2. Express in degrees the following angles measured in radians: (*a*) $\pi/2$; (*b*) $\pi/4$; (*c*) $\pi/6$; (*d*) $3\pi/2$; (*e*) 1; (*f*) 2; (*g*) 3.6; (*h*) 11.6.

[49.] Graphs of the Trigonometric Functions. In Fig. 26 let $OP_1P_2P_3$ be a circle with unit radius. Let OCP_1 be any angle at the center of the circle. Then P_1A_1 represents the sine of the angle OCP_1. Take OB equal in length to the circumference of the circle, *i.e.*, equal to 2π. Divide OB into thirty-six equal parts, and erect perpendiculars at each point of division. Each of these equal distances of the X-axis, OB, represents 10°, or $\pi/18$ radian. Upon each vertical line plot a point whose ordinate represents the sine of the corresponding angle. To do this divide the circumference of the circle into thirty-six equal parts, and join each to C, the center of the circle. (Only a few of these points, as P_1, P_2, P_3, P_4, P_5, are lettered in the drawing.) Project these points to the right upon the vertical line corresponding to the angle at the center of the

circle. Thus, the angle 50° determines the point P_2. P_2 projected upon the line passing through the point of the X-axis representing 50° gives P'_2. Then $P'_2A'_2 = P_2A_2 = \sin 50°$.

The curve $OKMNB$ (called the **sine curve**) is the graph of the equation $y = \sin x$, as x varies from 0 to 2π. It is readily seen that as x continues beyond this interval the curve is repeated for every addition of 2π to the X-axis.

The length of any portion of the X-axis represents an angle at the center of the circle expressed in circular units.

The graph of $y = 2 \sin x$ may be obtained from the curve in Fig. 26 by making all ordinates twice as long; the curve $y = 3 \sin x$ by making all ordinates three times as long; the curve $y = \frac{1}{2} \sin x$ by making all the ordinates one-half as long; the curve $y = \frac{1}{3} \sin x$ by making all ordinates one-third as long; etc.

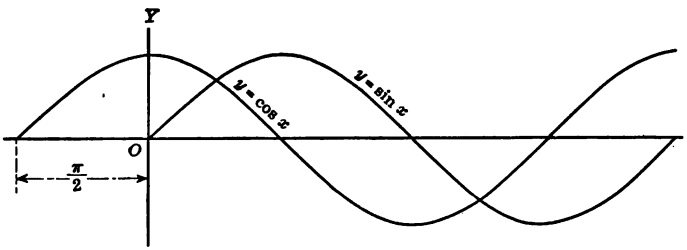


FIG. 27.

If the sine curve in Fig. 27 were translated to the left a distance $\pi/2$, its equation would be $y = \sin(x + \pi/2)$. Since the $\sin(x + \pi/2) = \cos x$, the translated curve in Fig. 27 is the graph for $y = \cos x$.

To construct the graph $y = \tan x$ we proceed by a method similar to that used in the construction of the sine curve.

Let C, Fig. 28, be the center of a circle of unit radius. Upon the X-axis lay off the distance OD equal to 2π, the circumference of the circle. Divide OD into thirty-six equal parts, and at each point of division erect a perpendicular. Let the Y-axis, $Y'OY$, be tangent to the circle. Beginning with the horizontal line draw thirty-six radii at intervals of 10°, extending them to the tangent

line $Y'OY$. Project the points of intersection with the tangent line upon the vertical line whose distance from the origin represents the particular angle at the center of the circle. Thus, OCP_1 is an angle of 30°. Project P_1 upon the third vertical line to the right of the Y-axis, and obtain P'_1, a point upon the graph of $y = \tan x$. When the angle is in the second or third quadrants the radius must be extended back through the center in order to intersect with the tangent $Y'OY$. We see that as the angle

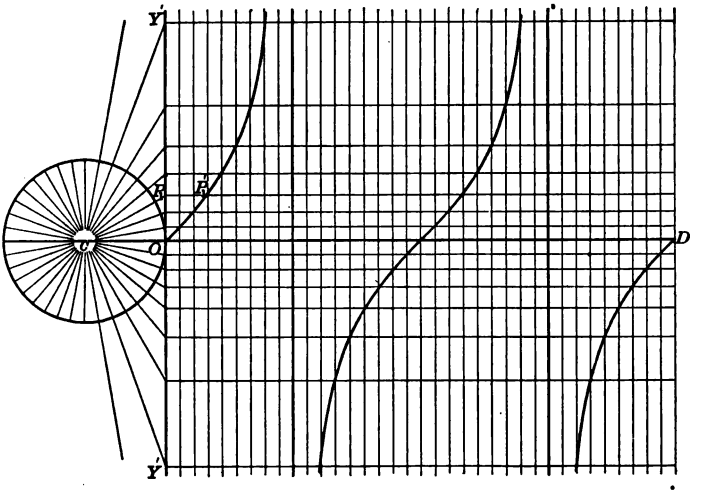


FIG. 28.—The tangent curve.

passes through 90° and 270° the tangent curve jumps from $+\infty$ to $-\infty$. The tangent curve extends to infinity in both y directions, and only a portion of it near the X-axis is represented in the figure. Curves for $y = \cot x$, $y = \sec x$, and $y = \csc x$ may be plotted from the tangent, cosine, and sine curves, respectively, by taking reciprocals of the lengths of the ordinates.

[50.] The Line Representation of the Trigonometric Functions. Let C, Fig. 29, be the center of a circle with *unit* radius. Let CO be a horizontal line and CD a vertical line. Let OY and DQ be

tangents. Let OCP be an angle α, at the center of the circle. Then, AP represents the sine of α; CA represents the cosine of α; OP_1 represents the tangent of α; DP_2 represents the cotangent of α; CP_1 represents the secant of α; and CP_2 represents the cosecant of α.

AO, which is equal to $1 - \cos \alpha$, is sometimes called the **versed sine** of α (**vers** α).

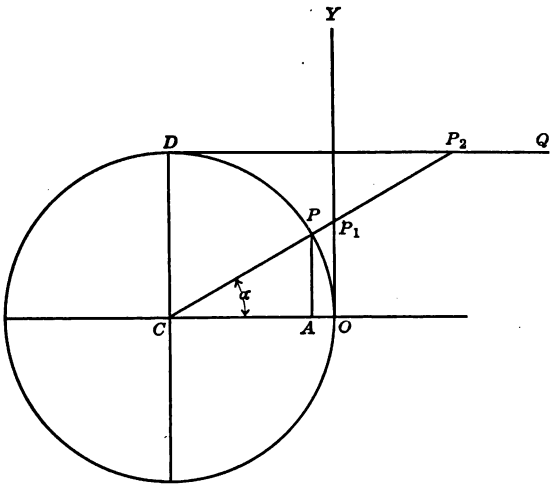


FIG. 29.

Exercises

1. Draw a curve for $y = \sin x$. Use 1.15 inches as the unit of length. This gives $2\pi = 7.2$ inches, approximately, or each 0.2 inch of the X-axis represents 10°.

2. Draw a curve for $y = \cos x$.

3. Draw a curve for $y = \tan x$.

4. Prove $(\sin \alpha)^2 + (\cos \alpha)^2 = 1$; or, as it is usually written, $\sin^2 \alpha + \cos^2 \alpha = 1$, where α is any angle.

HINT: In a triangle of reference for any angle, excluding 0°, 90°, 180°, and 270°, the square of the adjacent side plus the square of the opposite side equals the square of the hypotenuse. Note that this is true even though we consider the sides negative. For the angles

excepted above, prove the truth of the formula by direct substitution.

5. Prove $\sec^2 \alpha = 1 + \tan^2 \alpha$.

6. Prove $\csc^2 \alpha = 1 + \cot^2 \alpha$.

51. Trigonometric Tables. In preceding exercises we have calculated the numerical values of the trigonometric functions of a few angles; viz., 30°, 45°, 60°, etc. If the values of the functions of other angles were desired they could be found by drawing the angles with a protractor, constructing a triangle of reference, and from it scaling off the lengths of the sides and computing the ratios. This method of finding the functions would not only be slow, but the degree of accuracy would usually be unsatisfactory. A method has been devised for calculating the trigonometric functions to any number of decimal places, but it cannot be explained in this place.

A table of natural trigonometric functions, for every ten minutes from 0° to 90°, is given in Table XXII, pages 300–302. Since the cosecant and secant are, respectively, reciprocals of the sine and cosine, they need not be used in formulas, and their values are not usually published in a trigonometric table. If the angle is less than 45° the degrees and minutes of the angle are printed in the left-hand column and the names of the functions are printed at the top of the page. If the angle is greater than 45° the degrees and minutes are printed in the right-hand column and the names of the functions are printed at the bottom of the page. This arrangement of the table is possible through the fact that the cosine of an angle is the sine of its complement, and the cotangent is the tangent of its complement.

52. Graphic Table of Trigonometric Functions. From a sheet of polar coördinate paper, form *M*3, the sine and the cosine of any angle, and the tangent of an angle less than 45°, may be scaled off accurately to within two or three points in the third decimal place.

Upon this sheet concentric circles, 2/100 of a unit apart, are drawn. Every fifth circle is shown by a heavy line. Running from their common center radiating lines for every degree are drawn. Every tenth line is drawn heavy. Two additional circles are drawn each passing through the origin, the center of the con-

centric circles, and each having a diameter equal to unity. The circle with its center upon the Y-axis is called the sine circle, the other with its center upon the X-axis is called the cosine circle. At the right is a vertical scale from which may be read the tangent of any angle up to 45°. This scale is called the tangent scale, and its use is apparent at once. In explaining the method of scaling off the sines and cosines from the sine and cosine circles,

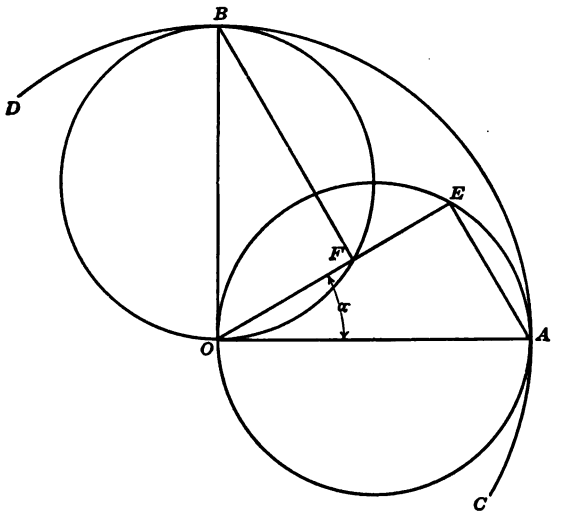


FIG. 30.

reference is made to Fig. 30. Let O be the center of the system of concentric circles, and let $CABD$ be the circle with radius equal to unity, and center at O. Let OEA be the cosine circle, and OFB the sine circle. Let AOE be any angle, say α. Draw AE and FB. OAE and OFB are right triangles. Then $OE = OA \cos \alpha$. But $OA = 1$. Therefore $OE = \cos \alpha$. Upon form $M3$ the length OD may be read off at once by means of the system of concentric circles.

Similarly, $OF = OB \sin OBF$. But, since $OB = 1$ and the angle $OBF =$ angle α, we have $OF = \sin \alpha$.

53. The Graphic Solution of the Right Triangle. Let A, B, and C be the vertices of any triangle; and let α, β, and γ be, respectively, the interior angles of the triangle at these vertices. Let the sides opposite the vertices A, B, and C be, respectively, a, b, and c. In a right triangle the right angle is marked γ, and the hypotenuse c.

The three sides together with the three angles are called the elements of the triangle. If any three of the six elements, excepting the three angles, of a possible triangle are given, the triangle may be constructed to scale, and the remaining three elements measured. The triangle is then said to be solved graphically.

Exercises

Solve graphically the following right triangles; record the time spent upon each problem.

1. $a = 173$, $b = 216$.
2. $b = 136$, $c = 527$.
3. $a = 210$, $c = 728$.
4. $a = 175$, $\alpha = 36°$.
5. $c = 516$, $\beta = 73°$.
6. $b = 172$, $\alpha = 47°$.

54. Solution of Right Triangles Analytically. Instead of solving a right triangle graphically we may solve it analytically. This method will be illustrated by solving two right triangles.

ILLUSTRATION 1: Given, $a = 216$ and $\alpha = 36°$; to find b, c, and β. We see at once that $\beta = 54°$, the complement of $36°$. To find c use the relation $c = \dfrac{a}{\sin \alpha}$, and to find b use the relation $b = \dfrac{a}{\tan \alpha}$.

From the table of the natural trigonometric functions,

$$\sin \alpha = 0.5878, \text{ and } \tan \alpha = 0.7265.$$

Substituting, $c = \dfrac{216}{0.5878} = 367.5$; and $b = \dfrac{216}{0.7265} = 297.3$.

To check the result, make use of the relation

$$c^2 - a^2 = b^2,$$

or

$$\sqrt{(c-a)(c+a)} = b.$$

$$c - a = 151.4.$$

$$c + a = 583.4.$$

$$\sqrt{(c-a)(c+a)} = 297.3 = b$$

This last value of b agrees with the value computed above. This not only checks the value of b, but also the value of a, which is used in computing the second value of b.

ILLUSTRATION 2: Given $c = 516.2$ and $a = 176.5$; to find b, α, and β.

Solution: $\sin \alpha = \frac{a}{c} = \frac{176.5}{516.2} = 0.3419$

From the table $\alpha = 19°.994$

whence $\beta = 70°.006$

$b = c \sin \beta = (516.2)(0.9397) = 485.1$

Check: $\sqrt{(c-a)(c+a)} = b,$

or $\sqrt{235{,}310.19} = b,$

or $485.1 = b,$

which checks all work.

Exercises

Solve the following right triangles. Use natural values of trigonometric functions taken from Table XXII. Perform the necessary multiplications and divisions by the arithmetical process, and hand in *all* numerical work. *Be sure to check your work.* Keep a record of time spent upon each problem.

[1.] $a = 173$, $b = 216$.
[2.] $b = 136$, $c = 527$.
[3.] $a = 210$, $c = 728$.
[4.] $a = 175$, $\alpha = 36°$.
[5.] $c = 516$, $\beta = 73°$.
[6.] $b = 172$, $\alpha = 47°$.

Instead of performing the multiplications and divisions by the lengthy arithmetical method, logarithms may be used. In the logarithmic solutions the natural values of the trigonometric functions are not used, because their logarithms may be taken directly from a logarithmic table. The two problems solved below illustrate the arrangement of the work. The student should go over the solutions and checks of these two problems very carefully, in order to become familiar with the use of a table of the logarithms of the trigonometric functions. Especial study should be given to the finding of the angles when the logarithms of the functions are given.

ILLUSTRATION 1: Given $a = 175.6$ and $b = 216.1$; to find α, β, and c.

Solution: $\tan \alpha = \frac{a}{b}$, $\beta = 180° - \alpha$, $c = \frac{b}{\sin \beta}$

$$\begin{aligned} \log a &= 2.2443 \\ \log b &= 2.3347 \\ \log \tan \alpha &= 9.9096 - 10 \\ \alpha &= 39° \, 4'.6 \\ \beta &= 50° \, 55'.4 \\ \log \sin \beta &= 9.8900 - 10 \\ \log c &= 2.4447 \\ c &= 278.44 \end{aligned}$$

Check:

$$\begin{aligned} c - a &= 102.84 \\ c + a &= 454.04 \\ \log (c - a) &= 2.0122 \\ \log (c + a) &= 2.6571 \\ \log b^2 &= 4.6693 \\ \log b &= 2.3347 \end{aligned}$$

The computed $\log b$ agrees with $\log b$ used above, which checks all of the work.

ILLUSTRATION 2: Given, $a = 176.3$, $\alpha = 37° \, 17'.4$; to find b, c, and β.

Solution: β (the complement of α) $= 52° \, 42'.6$. $b = a \tan \beta$, and $c = \frac{a}{\sin \beta}$.

$$\begin{aligned} \log a &= 2.2463 \\ \log \tan \beta &= 0.1183 \\ \log b &= 2.3646 \\ b &= 231.53 \\ \log \sin \beta &= 9.7824 - 10 \\ \log c &= 2.4639 \\ c &= 291.00 \end{aligned}$$

Check:

$$\begin{aligned} c - a &= 114.7 \\ c + a &= 467.3 \\ \log (c - a) &= 2.0596 \end{aligned}$$

$$\log (c + a) = 2.6696$$
$$\log b^2 = 4.7292$$
$$\log b = 2.3646$$

which agrees with **log b** above and thus checks the entire work.

This illustrative example was chosen so that the calculated value of the logarithm of b would agree with the value found in the table. This absolute agreement will, in general, not occur. For, in finding the second logarithm of b, calculated numbers are used which are correct only to four places, and the errors in these numbers, especially in $c-a$ will effect the calculated logarithm of b.

If the calculated logarithm differs from that taken from the table by only one or two in the fourth decimal place, the student may consider his work checked.

In checking the work it will be noticed that it is immaterial whether the **log a** or **log b** is recomputed. If the hypotenuse and a leg are given, the logarithm of the unknown leg should be recomputed. Otherwise the logarithm of the longer leg should be recomputed by the check formula. The reason for this is that it keeps the percent of error in the difference between the hypotenuse and the leg as small as possible.

Exercises

Solve the following right triangles using logarithms. Keep a record of the time spent upon each problem. *Check the work.*

1. $a = 173$, $b = 216$.
2. $b = 136$, $c = 527$.
3. $a = 210$, $c = 728$.
4. $a = 175$, $\alpha = 36°$.
5. $c = 516$, $\beta = 73°$.
6. $b = 172$, $\alpha = 47°$.

7. Having measured a distance of 281.6 feet in a direct horizontal line from the bottom of a tower, the angle of elevation of the top was found to be 24° 16′.6. Find the height of the tower.

8. A person on top of a tower 75 feet high observes the angle of depression of two objects on the horizontal plane, which are in line with the base of the tower, to be 37° 27′.6 and 25° 17′.8. Find the distance of each object from the base of the tower.

9. A tower stands by a river. A person on the opposite bank finds the angle of elevation of its top to be 50°. He recedes 40 yards

in a direct line from the tower, and finds the angle of elevation of the top to be 40°. Find the breadth of the river and the height of the tower.

10. A rope 100 feet long is fastened to the top of a building 40 feet high. Find the angle the rope is inclined, if its lower end just touches the ground.

11. From a balloon, which is directly over one town, is observed the angle of depression of another town, 12° 16′. The towns are 8.5 miles apart. Find the height of the balloon.

13. From a station A at the base of a mountain, its summit S is seen at an elevation of 43° 17′. After walking 5000 feet toward the summit, up a plane making an angle of 28° 13′ with the horizontal, to another station, B, the angle ABS was found to be 137° 17′. Find the height of the mountain.

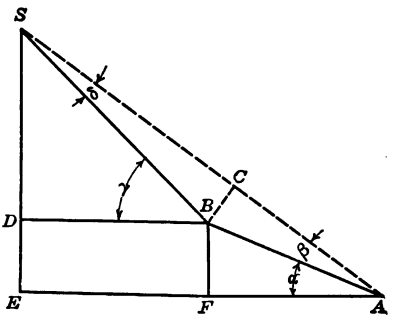


FIG. 31.

HINT: In solving any problem, first build up a formula giving the unknown number. As far as possible put this formula in a form suitable for logarithmic computation. This will be illustrated by building up a formula for x, the height of the mountain. In Fig. 31, $SE = x$ is the required height. $BA = 5000$ feet; $\alpha = 28° 13'$ and the angles β, γ, and δ can be computed from the given data. BC is perpendicular to AS.

$$BC = 5000 \sin \beta$$

$$BS = \frac{BC}{\sin \delta} = \frac{5000 \sin \beta}{\sin \delta}$$

$$DS = BS \sin \gamma$$

$$= 5000 \sin \beta \frac{\sin \gamma}{\sin \delta}$$

$$DE = BF = 5000 \sin \alpha$$

$$x = DS + DE = 5000 \sin \beta \frac{\sin \gamma}{\sin \delta} + 5000 \sin \alpha$$

$$= 5000 \left[\frac{\sin \beta \sin \gamma}{\sin \delta} + \sin \alpha \right]$$

Compute β, γ, and δ, and substitute in the formula. Compute the value of the first term within the parenthesis by using logarithms.

13. From a point in a window in the same horizontal plane with the bottom of a steeple, the angle of elevation of the top of the steeple is 43° 16′. From another window, 18 feet below the first, the angle of elevation of the top of the steeple is 52° 8′. Build up a formula giving the height of the steeple. Find the height.

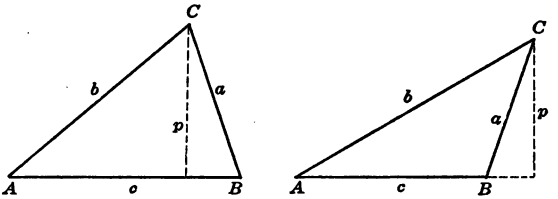


FIG. 32.

55. The Law of Sines. Let ABC, Fig. 32, be any triangle. Let p be a perpendicular let fall from C upon c. Then

$$p = a \sin \beta$$

and

$$p = b \sin \alpha.$$

Hence

$$a \sin \beta = b \sin \alpha,$$

or

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta}.$$

In a similar way we may show that

$$\frac{a}{\sin \alpha} = \frac{c}{\sin \gamma}.$$

Hence

$$\frac{\mathbf{a}}{\mathbf{\sin \alpha}} = \frac{\mathbf{b}}{\mathbf{\sin \beta}} = \frac{\mathbf{c}}{\mathbf{\sin \gamma}},$$

which is known as the **law of sines.**

Exercises

1. Suppose we are required to determine the distance from a given point A to an inaccessible object B, Fig. 33. Let C be any other

convenient accessible point. Set a transit over A and measure the angle BAC. Set the transit over C and measure the angle BCA. Measure the distance AC. Solve for AB, using the law of sines.

Data:

$$b = 500 \text{ feet}$$
$$\alpha = 101° 13' 6$$
$$\gamma = 42° 28'.7$$

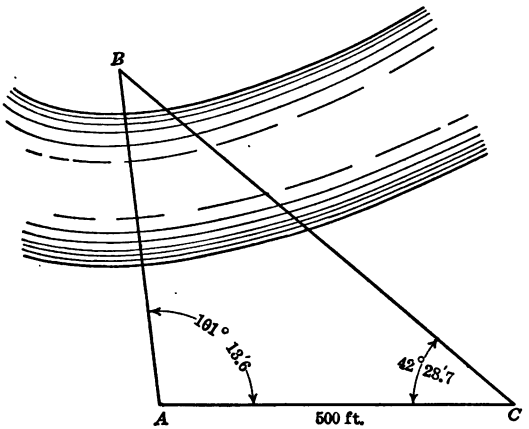


FIG. 33.

To find $c = AB$, use the formula

$$\frac{c}{\sin \gamma} = \frac{b}{\sin \beta},$$

or

$$c = \frac{b \sin \gamma}{\sin \beta}.$$

The value of β may be found by subtracting the sum of α and γ from 180°, for, from geometry, the sum of the three interior angles of a triangle is equal to 180°.

2. Same as exercise 1, with the following data:

$$b = 375.6 \text{ feet}$$
$$\alpha = 27° 42'.7$$
$$\gamma = 113° 17'.6.$$

HINT: Sin (113° 17'.6) = sin (66° 42'.4.) = cos (23° 17'.6)

56. The Law of Cosines. Let ABC, Fig. 34, be any triangle. Let p be a perpendicular let fall from B upon b.

$$c^2 = p^2 + (b - k)^2$$

if γ is acute (Fig. 34, a).

$$c^2 = p^2 + b^2 - 2bk + k^2.$$

Since $p^2 + k^2 = a^2$, we have,

$$c^2 = a^2 + b^2 - 2bk$$

and since $k = a \cos \gamma$, we have

$$c^2 = a^2 + b^2 - 2ab \cos \gamma.$$

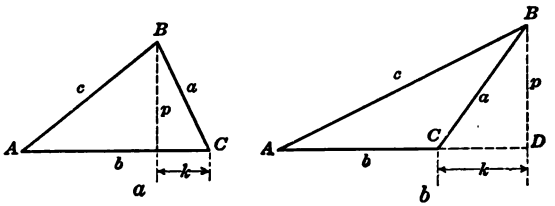


FIG. 34.

If γ is obtuse (Fig. 34, b),

$$c^2 = p^2 + (b + k)^2$$
$$c^2 = p^2 + b^2 + 2bk + k^2$$
$$c^2 = a^2 + b^2 + 2kb.$$

But, since $k = a \cos DCB = -a \cos \gamma$, we obtain the same expression for c as above,

$$\mathbf{c^2 = a^2 + b^2 - 2ab \cos \gamma},$$

which is known as the **law of cosines.**

Exercises

1. Suppose we wish to determine the distance between two points A and B on opposite sides of a building, Fig. 35. Choose any convenient point, C, and measure AC, CB, and the angle ACB.

Data:

$$a = 137.6 \text{ feet}$$
$$b = 186.5 \text{ feet}$$
$$\gamma = 76° 54'.6.$$

To find c use formula

$$c^2 = a^2 + b^2 - 2ab \cos \gamma.$$

2. Same as exercise 1, with the following data:

$a = 378.6$ feet.
$b = 276.5$ feet.
$\gamma = 112° 27'.8$.

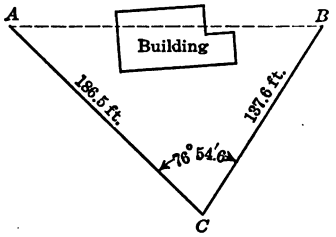


FIG. 35.

57. The Law of Tangents. Let ABC, Fig. 36, be any triangle. With C as center and with CA as radius describe an arc cutting BC at E and BC produced at D. Draw AE and AD. EAD is a right triangle with the angle AEC equal to $\frac{\alpha + \beta}{2}$. The angle EAB is equal to $\frac{\alpha - \beta}{2}$. Draw EK perpendicular to AE.

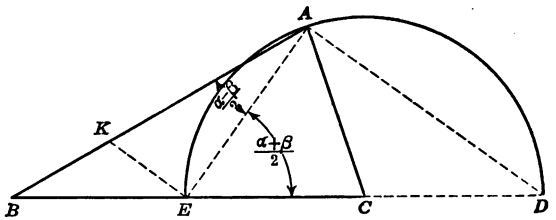


FIG. 36.

The triangles BEK and BAD are similar. Hence

$$\frac{BD}{BE} = \frac{AD}{EK} = \frac{AD}{AE}\frac{AE}{EK} = \tan\frac{\alpha + \beta}{2}\cot\frac{\alpha - \beta}{2}$$

Since $BD = a + b$, and $BE = a - b$,

$$\frac{\mathbf{a + b}}{\mathbf{a - b}} = \frac{\tan\ \frac{1}{2}(\alpha + \beta)}{\tan\ \frac{1}{2}(\alpha - \beta)}, \tag{1}$$

which is known as **the law of tangents.** Since $\alpha + \beta + \gamma = 180°$,

$$\frac{\alpha + \beta}{2} = 90° - \frac{\gamma}{2}.$$

The law of tangents may be used instead of the law of cosines for solving triangles in which two sides and the included angle are given.

ILLUSTRATION: Given

$$a = 337.6 \text{ feet}$$
$$b = 213.5 \text{ feet}$$
$$\gamma = 27° 16'$$

to find c, α and β.

Solution:

$$\tan \tfrac{1}{2}(\alpha - \beta) = \frac{a - b}{a + b} \tan \tfrac{1}{2}(\alpha + \beta)$$
$$a - b = 124.1$$
$$a + b = 551.1$$
$$\tfrac{1}{2}(\alpha + \beta) = 76° 22'$$
$$\log (a - b) = 2.0938$$
$$\log \tan \tfrac{1}{2}(\alpha + \beta) = 0.6152$$
$$\log (a + b) = 2.7413$$
$$\log \tan \tfrac{1}{2}(\alpha - \beta) = \overline{9.9677} - 10$$
$$\tfrac{1}{2}(\alpha - \beta) = 42° 52'$$
$$\alpha = 119° 14'$$
$$\beta = 33° 30'$$

Find the value of c by using the law of sines,

$$c = \frac{b \sin \gamma}{\sin \beta}$$
$$\log b = 2.3294$$
$$\log \sin \gamma = 9.6610 - 10$$
$$\log \sin \beta = 9.7419 - 10$$
$$\log c = \overline{2.2485}$$
$$c = 177.2$$

Exercises

Solve exercises 1 and 2, § 56, by using the law of tangents.

58. The Addition Formulas. Let ABC, Fig. 37, be any triangle. Let ACD be the exterior angle formed by producing the side BC. Draw CK perpendicular to AB.

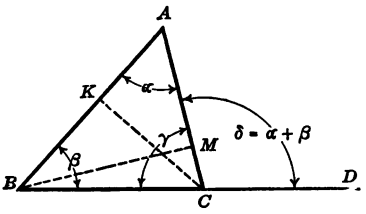


FIG. 37.

The law of sines gives,

$$\sin \gamma = \frac{AB \sin \alpha}{BC}. \tag{1}$$

Since $\alpha + \beta = 180° - \gamma$, $\sin(\alpha + \beta) = \sin \gamma$. Equation (1) may then be written

$$\begin{aligned} \sin(\alpha + \beta) &= \frac{AB \sin \alpha}{BC} \\ &= \sin \alpha \frac{BK}{BC} + \sin \alpha \frac{AK}{BC}, \\ &= \sin \alpha \cos \beta + \frac{AK}{AC} \frac{CK}{BC}, \end{aligned}$$

or

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta. \tag{2}$$

This formula gives the sine of the sum of two angles, provided the sum is less than 180°. We shall show that this limitation may, however, be removed. Let $\alpha + \beta$ be greater than 180° but less than 360°. We then have $\alpha + \beta = 180° + \epsilon + \beta$, where $\epsilon + \beta$ is less than 180°.

$$\sin(\alpha + \beta) = \sin[180° + (\epsilon + \beta)] = -\sin(\epsilon + \beta)$$

By formula (2)

$$\sin(\epsilon + \beta) = \sin \epsilon \cos \beta + \cos \epsilon \sin \beta$$

and since $\epsilon = \alpha - 180°$,

$$\sin(\alpha + \beta) = -[\sin(\alpha - 180°)\cos\beta + \cos(\alpha - 180°)\sin\beta],$$

or

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta,$$

which shows that formula (2) is true if $\alpha + \beta$ is less than 360°.

Similarly it may be shown that formula (2) is true for any value of the sum, $\alpha + \beta$.

To prove a similar formula for $\sin(\delta - \alpha)$, drop a perpendicular, BM, Fig. 37, from B upon AC:

$$\sin\beta = \sin(\delta - \alpha)$$

By the law of sines

$$\sin\beta = \frac{AC\sin\alpha}{BC}$$

$$= \frac{AM + MC}{BC}\sin\alpha$$

$$= \frac{BM}{BC}\frac{AM}{BA} + \frac{MC}{BC}\sin\alpha,$$

or

$$\sin(\delta - \alpha) = \sin\gamma\cos\alpha + \cos\gamma\sin\alpha \qquad (3)$$

Since $\sin\gamma = \sin\delta$ and $\cos\gamma = -\cos\delta$, equation (3) becomes

$$\sin(\delta - \alpha) = \sin\delta\cos\alpha - \cos\delta\sin\alpha \qquad (4)$$

Formula (4) gives the sine of the difference of two angles provided neither angle exceeds 180°. The student will show that this limitation can be removed in a way similar to the way in which it was removed from formula (2) above.

Formulas (2) and (4) are usually written together as,

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta. \qquad (5)$$

Since $\cos(\alpha \pm \beta) = \sin[(90° - \alpha) \mp \beta]$,

formula (5) gives

$$\cos(\alpha \pm \beta) = \sin(90° - \alpha)\cos\beta \mp \cos(90° - \alpha)\sin\beta,$$

or

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta. \qquad (6)$$

Formulas (5) and (6) are called the **addition formulas** for the sine and cosine.

If $\beta = \alpha$, formula (6) gives

$$\cos(\alpha + \alpha) = \cos\alpha\cos\alpha - \sin\alpha\sin\alpha,$$

or

$$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha, \tag{7}$$

or

$$\cos 2\alpha = 1 - 2 \sin^2 \alpha, \tag{8}$$

since $\cos^2 \alpha = 1 - \sin^2 \alpha$.

$$\cos 2\alpha = 2 \cos^2 \alpha - 1 \tag{9}$$

When $\beta = \alpha$, formula (2) becomes

$$\sin (\alpha + \alpha) = \sin \alpha \cos \alpha + \cos \alpha \sin \alpha$$

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha. \tag{10}$$

Exercises

1. Find without using the tables, the exact value of sin 75°, and of cos 75°.

HINT: $75° = 45° + 30°$.

2. Find without using the tables, the exact value of sin 15°; and of cos 15°.

HINT: $15° = 45° - 30°$.

3. Show that $\sin \frac{x}{2} = \sqrt{\frac{1 - \cos x}{2}}$.

HINT: Let $2\alpha = x$ in formula (8).

4. Show that $\cos \frac{x}{2} = \sqrt{\frac{1 + \cos x}{2}}$.

HINT: Let $2\alpha = x$ in formula (9).

5. Show that

$$\tan \frac{x}{2} = \sqrt{\frac{1 - \cos x}{1 + \cos x}}$$

$$= \frac{1 - \cos x}{\sin x}$$

$$= \frac{\sin x}{1 + \cos x}.$$

HINT: Use results of exercises 4 and 5.

[59.] Area of Triangle by Drawing to Scale. In **§ 41**, the areas of triangles, with the three sides given, were found both graphically and analytically.

In the following exercises construct the triangle to scale. Drop a perpendicular from any vertex upon the opposite side. Measure the length of this perpendicular, and multiply this length by the length of the side upon which it was dropped. One-half of this product is the area of the triangle. In drawing the triangle use a convenient scale, just small enough to admit drawing upon a sheet of paper, $8\frac{1}{2} \times 11$ inches. Always mark in words upon the drawing the scale used, as "scale 1 inch = 1 foot"; "scale 1/2 inch = 10 rods"; "scale 1/4 inch = 20 rods," etc.

Exercises

Find the area of each of the following triangles. Record time spent upon each problem. Place but one problem upon a sheet of paper.

1.	$a = 175.6$ rods;	$b = 276.5$ rods;	$\beta = 27° 35'$.
2.	$a = 267.75$ feet;	$\beta = 37° 45'$;	$\gamma = 43° 17'.6$.
3.	$b = 262.5$ feet;	$a = 178.4$ feet;	$\beta = 34° 47'.5$.
4.	$b = 175.37$ rods;	$c = 216.82$ rods;	$\alpha = 27° 47'$.
5.	$b = 73.78$ feet;	$c = 53.62$ feet;	$\alpha = 136° 43'.6$.
6.	$b = 100.37$ feet;	$\alpha = 27° 31'.6$;	$\beta = 73° 6'.7$.

60. Area of Triangle by Formula. *Case I. Two sides and the included angle given.*

Let ABC be any triangle with b, c and α given. From C draw p perpendicular to the side AB. Then

$$\text{area} = \tfrac{1}{2}pc.$$

Since $p = b \sin \alpha$,

$$\textbf{area} = \tfrac{1}{2}bc \sin \alpha.$$

ILLUSTRATIONS: Find the area of the triangle having given

$$b = 172.36 \text{ feet}$$
$$c = 103.27 \text{ feet}$$
$$\alpha = 27° 17'.6.$$
$$\log b = 2.2364$$
$$\log c = 2.0140$$
$$\log \sin \alpha = 9.6614 - 10$$
$$\log \tfrac{1}{2} = 9.6990 - 10$$
$$\log \text{area} = 3.6108$$
$$\text{area} = 4081 \text{ square feet}$$

Exercises

Find the area of each of the following triangles:

1. $a = 173.6$ rods $b = 263.8$ rods $\gamma = 73° 47'.6$.
2. $a = 278.3$ rods $c = 172.7$ rods $\beta = 21° 21'.7$.
3. $b = 100.1$ rods $c = 216.5$ rods $\alpha = 127° 26'.3$.

Case II: One side and two angles given.

If two angles are given the third can be computed by subtracting their sum from 180°. Let ABC be any triangle with c, α and β given. From C drop a perpendicular, p, upon the side c. Let M be the foot of this perpendicular.

Then

$$AM = p \cot \alpha$$
$$BM = p \cot \beta$$

and by adding

$$\begin{aligned} c &= p(\cot \alpha + \cot \beta) \\ &= p\left[\frac{\cos \alpha}{\sin \alpha} + \frac{\cos \beta}{\sin \beta}\right] \\ &= p\left[\frac{\sin \beta \cos \alpha + \cos \beta \sin \alpha}{\sin \alpha \sin \beta}\right] \\ &= p \frac{\sin (\alpha + \beta)}{\sin \alpha \sin \beta}. \end{aligned}$$

Then

$$p = \frac{c \sin \alpha \sin \beta}{\sin (\alpha + \beta)}.$$

Since $\text{area}, A = \frac{1}{2}pc,$

$$A = \frac{c^2 \sin \alpha \sin \beta}{2 \sin (\alpha + \beta)}.$$

ILLUSTRATION: Find the area of the triangle having given $c = 263.75$ feet, $\alpha = 33° 17'.6$, and $\beta = 73° 21'.6$.

$$\begin{aligned} \alpha + \beta &= 106° 39'.2 \\ \log c &= 2.4212 \\ \log c^2 &= 4.8424 \\ \log \sin \alpha &= 9.7395 - 10 \\ \log \sin \beta &= 9.9814 - 10 \\ \log \tfrac{1}{2} &= 9.6990 - 10 \\ \log \sin (\alpha + \beta) &= 9.9816 - 10 \\ \log \text{area} &= 4.2807 \\ \text{area} &= 1909 \text{ square feet} \end{aligned}$$

The log sin $(\alpha + \beta)$, or the log cos 16° 39′.2, is subtracted from the sum of the four preceding logarithms. The subtraction is performed digit by digit, as the addition progresses from right to left.

Exercises

Find the area of each of the following triangles. Record time spent upon each exercise.

1.	$c = 216.3$ feet;	$\alpha = 24° 16'.7$;	$\beta = 61° 15'.6$.
2.	$a = 372.6$ feet;	$\alpha = 37° 27'.7$;	$\beta = 81° 21'.6$.
3.	$b = 87.75$ feet;	$\beta = 16° 37'.6$;	$\gamma = 63° 21'.7$.
4.	$c = 217.75$ feet;	$\alpha = 93° 17'.5$;	$\beta = 21° 37'.6$.
5.	$\alpha = 126° 17'.6$;	$\gamma = 25° 57'.3$;	$b = 200.37$ feet.

[*Case III.*] *Given two sides and an angle opposite one of them.* Let ABC be any triangle. Let a, c, and α be given. Drop a perpendicular, p, from B upon AC. Let M be the foot of the perpendicular.

$$\begin{aligned} MA &= c\cos\alpha, \\ p &= c\sin\alpha \\ MC &= \sqrt{a^2 - p^2} = \sqrt{a^2 - c^2\sin^2\alpha} \\ AC &= AM \pm MC \end{aligned}$$

The minus sign is used if γ is obtuse. Substituting for AM and MC,

$$AC = c\cos\alpha \pm \sqrt{a^2 - c^2\sin^2\alpha}.$$

Then, since area $= \frac{1}{2}pAC$,

$$\text{area} = \tfrac{1}{2}c\sin\alpha\left[c\cos\alpha \pm \sqrt{a^2 - c^2\sin^2\alpha}\right]$$

$$\text{area} = \tfrac{1}{2}c^2\sin\alpha\left[\cos\alpha \pm \sqrt{(a/c - \sin\alpha)(a/c + \sin\alpha)}\right]$$

If $a/c < \sin\alpha$, the quantity under the radical sign is negative, which means that in this problem there is no real area. From a figure it will be seen that this is the impossible case studied in geometry.

If $a/c = \sin\alpha$ we have a right triangle, and the formula reduces to: area $= \frac{1}{2}c^2\sin\alpha\cos\alpha$, which is as it should be.

If γ is acute use the + sign before the radical; if obtuse use the − sign.

If $a > c$, and if α is the *interior* angle of the triangle, there is but one solution given by the formula with the + sign before the radical.

ILLUSTRATION: Find the area of the triangle if $c = 216.7$ feet, $a = 187.56$ feet, and $\alpha = 37° 15'.6$, and γ obtuse.

$$\begin{aligned}
\log a &= 2.2730 \\
\log c &= 2.3359 \\
\log a/c &= 9.9371 - 10 \\
a/c &= 0.8652 \\
\sin \alpha &= 0.6054 \\
a/c - \sin \alpha &= 0.2598 \\
a/c + \sin \alpha &= 1.4706 \\
\log(a/c - \sin \alpha) &= 9.4147 - 10 \\
\log (a/c + \sin \alpha) &= 0.1675 \\
\text{sum} &= 9.5822 - 10 \\
\tfrac{1}{2} \text{ sum} &= 9.7911 - 10 \\
\sqrt{(a/c)^2 - \sin^2 \alpha} &= 0.6181 \\
\cos \alpha &= 0.7958^1 \\
\cos \alpha - \sqrt{(a/c)^2 - \sin^2 \alpha} &= 0.1777 \\
\log 0.1777 &= 9.2497 - 10 \\
\log \sin \alpha &= 9.7821 - 10 \\
\log c^2 &= 4.6718 \\
\log \tfrac{1}{2} &= 9.6990 - 10 \\
\log \text{area} &= 3.4026 \\
\text{area} &= 2527 \text{ square feet.}
\end{aligned}$$

Exercises

Find the areas in acres, of each of the following triangles. Record time spent upon each problem.

1. $c = 317.62$ rods, $a = 217.75$ rods, $\alpha = 27° 41'.7$, and γ is obtuse.

2. $c = 267.7$ rods, $a = 298.6$ rods, $\alpha = 87° 8'.6$, and γ is acute.

Case IV. Three sides given. This case has already been discussed in § **41.**

61. Graphic Method of Constructing the Curve of Intersection of a Right Circular Cylinder with a Plane. In exercise 3, § **42,** there was described a method of constructing the curve of intersection of a right circular cylinder and a plane making an angle of 60° with the axis of the cylinder. A graphic method will now be

[1] In taking out the cosine or the logarithm of the cosine of an angle, the correction, if any, must be subtracted, because, in the first quadrant, the cosine decreases as the angle increases.

described for the construction of this curve, not only when the angle is 60°, but for any given angle.

Let $BQCB'$, Fig. 47, be the normal cross section of the cylinder,[1] and let $BPAB'$ be the section made by a plane making an angle α with the normal cutting plane. From the explanation given in exercise 3, § **42**, it is readily seen that

$$RP = \frac{RQ}{\cos \alpha} = RQ \sec \alpha.$$

Draw the circle $BQCB'$, Fig. 38, with radius OB equal to the radius of the cylinder. Extend the diameter BOB' to some convenient point O'. Consider $O'B$ and $O'Y'$ a set of rectangular

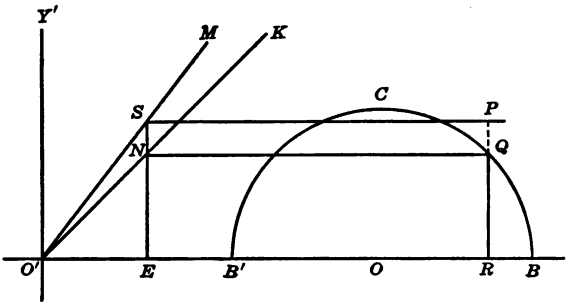


FIG. 38.

coördinate axes. Through O' draw the 45° line $O'K$ and the line $O'M$ whose slope is equal to the value of sec α. Draw QN parallel to $O'B$ intersecting $O'K$ at N. Draw NS parallel to $O'Y'$ intersecting $O'M$ at S. Draw SP parallel to $O'B$ intersecting RQ, produced, at P. P is a point on the curve to be drawn. For, since $O'E = EN = RQ$, and since $ES = RP = O'E \sec \alpha$, $RP = RQ \sec \alpha$.

In this manner any number of points may be located upon the curve.

In the above construction the horizontal and vertical lines need not actually be drawn; it is sufficient to locate only the points N and S. If the drawing is done on squared paper the horizontal

[1] A normal cross section is a section made by a cutting plane perpendicular to the axis of the cylinder.

and vertical lines of the paper will be of assistance; if the drawing is done on plain paper the T-square and triangle should be used.

Exercises

Construct the curve called for in exercise 3, § **42**, by the method of this section. Construct the complete curve on a heavy sheet of paper, cut out the inner portion and fold to form a model of the saddle.

62. The Pattern for a Right Circular Cylindrical Surface Cut by a Plane. In the preceding section a method was given for laying out the saddle for a circular ventilator. A method will now be

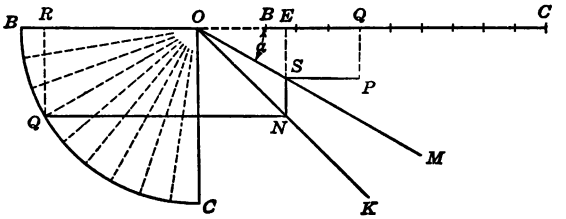


FIG. 39.

described for laying out a pattern for cutting the sheet iron which when rolled up into a cylindrical surface will give the flue fitting upon this saddle. The length of this pattern will be equal to the circumference of the cylinder.

Draw the line *BQC*, Fig. 39, equal in length to one-fourth the circumference of the cylinder. This line corresponds to the arc *BQC*, Fig. 47, *i.e.*, if the sheet of paper upon which *BC* is drawn is rolled up into a cylindrical surface such that lines perpendicular to *BC* become the elements of the cylinder, *BC* becomes the arc *BQC* of Fig. 47. From points of *BC*, such as *Q*, perpendiculars must be laid off equal to *QP*, Fig. 47. From Fig. 47, it is apparent that the distance

$$QP = RQ \tan \alpha$$

where α is the angle between the normal and inclined cutting planes.

Extend *CB* to some convenient point *O*. With *O* as center and with a radius equal to the radius of the cylinder, describe the arc

BQC. Through O draw the 45° line OK, and the line OM making the angle MOC equal to α. By means of a protractor divide the arc BQC into a number of equal parts (nine for convenience). Divide the line BC into the same number of equal parts. From each point of division of the arc, as Q, draw a horizontal line intersecting OK at N. Through N draw a vertical line intersecting OM at S. Through S draw a horizontal line intersecting at P the perpendicular erected at Q of the line BC. P is a point upon the desired curve. For, since $QR = NE = OE$, and since $PQ = SE = OE \tan \alpha$,

$$PQ = RQ \tan \alpha.$$

Any number of points may be located in this way and a smooth curve drawn through them. The pattern thus obtained may be used four times, giving the complete pattern.

Exercises

1. Construct a pattern by the method explained above using the dimensions in the exercise of the preceding section.

HINT: This drawing will require a sheet of paper larger than $8\frac{1}{2} \times 11$ inches.

2. An elbow is formed by joining two right circular cylinders of equal radii in such a way that their axes intersect at right angles. Lay out a pattern for the surfaces.

63. Pattern for a Conical Roof. A pattern is to be drawn for a conical roof, Fig. 40. It is apparent that the pattern may be made by cutting out a circular sector from a circular sheet of paper of radius r.

If R is given,

$$r = R \sec \alpha.$$

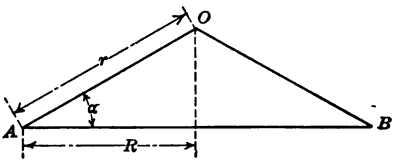


FIG. 40.

The circumference of the circular sheet of paper from which the circular sector is cut, Fig. 41, is then $2\pi R \sec \alpha$. The circumference of the lower edge of the roof is $2\pi R$. Hence the length of the arc

$$DE = 2\pi R \sec \alpha - 2\pi R$$
$$= 2\pi R(\sec \alpha - 1).$$

Then the angle EOD expressed in radians is

$$\beta = \frac{2\pi R(\sec \alpha - 1)}{r}$$
$$= \frac{2\pi R(\sec \alpha - 1)}{R \sec \alpha}$$
$$= 2\pi(1 - \cos \alpha).$$

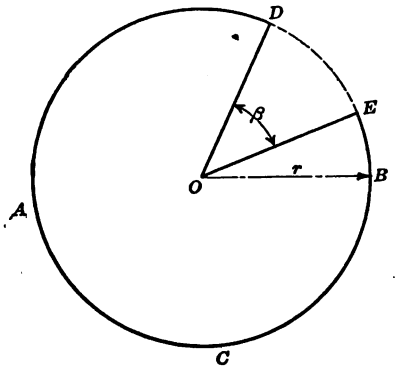


FIG. 41.

Exercises

1. Construct a pattern for a conical roof if α, Fig. 40, is 30°, and if R is 20 percent greater than the radius used in exercise 1 of the preceding section.

2. Build up suitable formulas for a pattern for the convex surface of a tank in the form of a frustum of a right circular cone, if the radius of the larger base is R, of the smaller base is r, and if the depth is H, Construct a pattern (reduced) if $R = 5$, $r = 4$, and $H = 3$.

Miscellaneous Exercises

1. The roof rises from the adjacent sides of a rectangular house at an angle of 30°. Find the angle which the corner of the roof makes with the horizontal. Find the length of this edge measured from the plate to the ridge, if the building is 24 × 32 feet. What is the length of the ridge of the roof?

2. The height of a butte is desired. At a point B, the angle of

elevation of the base of a tree, A, standing on the top of the butte was measured and found to be 63° 18′.7. At a second point, C, in the same vertical plane with AB, at the same elevation with, and 200 feet from, the point B, the angle of elevation of A was found to be 32° 7′.8. Build up a formula giving x, the elevation of A above B; and y, the horizontal distance of A from B.

3. Same as exercise 2, excepting that instead of ABC being in the same vertical plane the angle between the vertical planes through A and B, and B and C is 163° 5′.6.

4. Same as exercise 2, excepting that the point C is 10 feet higher than the point B.

5. Same as problem 3, excepting that instead of B and C being on a level, the point C is 10 feet lower than the point B.

6. A 100-foot tape hangs down the face of a vertical cliff. The zero mark is at A, and the 100 mark at B. At a point C in the plane below the cliff, the angle of elevation of A is 43° 30′.7, and of B is 36° 20′.7. Find the elevation of the point A above C.

7. A and B are two points on the face of a vertical cliff. At a point C the vertical line AB subtends an angle of 7° 10′; *i.e.*, the angle $BCA = 7°\ 10'$. At a point D it subtends an angle of 6° 7′. If A, B, C and D are in the same vertical plane and if C and D are 200 feet apart at the same elevation, find the elevation of A above C, and the horizontal distance of A from C.

8. Same as exercise 7, but let D be 5 feet lower than C.

9. Same as exercise 7, but let the vertical plane through CD make an angle of 137° 10′.7 with the vertical plane through AC.

10. Same as exercise 9, but let D be 6 feet lower than C.

11. A, B, C, D, E, and F are the vertices of a six-sided field. Find the area in acres if:

$AB = 162.75$ rods.
$BC = 96.63$ rods.
$CD = 86.73$ rods.
$DE = 100.75$ rods.
$EF = 101.62$ rods.
$FA = 98.76$ rods.
$AC = 200.60$ rods.
$AD = 156.73$ rods.
$AE = 200.00$ rods.

12. The line AB runs north and south. The line AC makes an angle of 52° 8′.6 with AB. Locate the line BC perpendicular to AB so that the area ABC shall be 1 acre.

13. Same as exercise 12, excepting that the angle ABC is to be 29° 17′.6 instead of 90°.

14. The area, in acres, of a four-sided plot of land with vertices M, N, O, and P, is desired. The side MP crosses a low swamp so that it can not be chained. The sides MN, NO, and OP are measured and found to be, respectively, 798.8 feet, 7103.7 feet and 2181.1 feet. The angles PMN and NOP are measured and found to be, respectively, 76° 47′.6, and 126° 10′.2.

HINT: Draw the diagonal NP and find the area of each triangle.

15. Same as exercise 14, but instead of measuring the angle PMN the angle MNP was measured, and found to be 63° 42′.6.

HINT: Compute the length of NP as a side of the triangle NOP.

16. Same as exercise 14, but angle ONM was measured instead of angle, PMN and found to be 127° 0′.3.

HINT: Compute the size of the angle ONP as an angle of the triangle NOP.

17. M, N, O, and P are the vertices of a four-sided field. $NO = 3172.6$ feet, $\angle MNP = 43° 21'.3$, $\angle PNO = 51° 2'.1$, $\angle NOM = 55° 6'.5$, and $\angle MOP = 47° 52'.6$. Find the area of the field in acres.

HINT: Calculate MN and angle MNO. Find the area of the triangles NOP and MNP.

18. Find the volume, in cubic yards, of a right circular cylindrical shell of uniform thickness. To make the problem specific, let us find the number of yards of concrete in the vertical wall of a circular concrete silo, assuming, first, that there are no openings. Let R be the outer radius, d the thickness of the wall and H the height, all measured in feet. From the horizontal cross section of the silo, it is seen that the area of the cross section of the wall is the difference of the areas of the two circles, one with radius R and the other with radius $R - d$. This area $A = \pi R^2 - \pi(R - d)^2 = \pi[R^2 - (R - d)^2]$. The quantity within the parentheses is the difference between two squares, and may be resolved into the product of two factors, the one the sum and the other the difference of the numbers. Then $A = \pi(2R - d)(d)$. This area multiplied by the height and divided by 27 gives the volume, V, of the wall in cubic yards. Then

$$V = \frac{\pi(2R - d)(d)(H)}{27}$$

a formula suitable for logarithmic computation.

ILLUSTRATION: Suppose the outside diameter of a silo is 13 feet,

the thickness of the wall 7 inches (0.5833 feet, from Table XVII), and height 35 feet. Then

$$\begin{aligned} H &= 35 \\ d &= 0.5833 \\ 2R - d &= 12.4167 \\ \log H &= 1.5441 \\ \log d &= 9.7659 - 10 \\ \log (2R - d) &= 1.0940 \\ \log \pi &= 0.4971 \\ \log 27 &= 1.4314 \\ \hline \log V &= 1.4697 \\ V &= 29.49 \text{ cubic yards.} \end{aligned}$$

If only a single computation were to be made, the formula for volume could be built up using the data directly. For the above measurements:

$$\begin{aligned} \pi(6.5)^2 &= \text{area of larger circle} \\ \pi(6.5 - 0.5833)^2 &= \text{area of inner circle} \\ \pi(6.5^2 - 5.9167^2) &= \pi(6.5 + 5.9167)(6.5 - 5.9167) \\ &= \pi(12.4167)(0.5833) \\ &= \text{area of cross section} \end{aligned}$$

Then

$$\begin{aligned} V &= 35\pi(12.4167)(0.5833) \div 27 \\ &= \text{volume in cubic yards.} \end{aligned}$$

If several sets of measurements were given, and if volumes corresponding to all were to be computed, it would be desirable to derive first the general literal formula and then substitute the given numerical values. The example explained above illustrates how much easier it is to effect the transformations when letters rather than numerical values are used.

The student is reminded that in general it is advisable to represent the known and unknown numbers by letters and build up, if possible, a formula for each unknown in terms of known numbers before making numerical substitutions.

19. Find the volume in cubic yards of a right circular cylindrical shell, if the outer radius is 7.25 feet, the thickness 8 inches, and the length 28.5 feet.

20. In exercise 18, the volume of the opening must be deducte from the total volume of the wall. Let the sides of the opening be two vertical planes passing, if extended, through the axis of the

cylinder.[1] Fig. 42 represents a cross section of the opening. Let S be the width of the opening on the outside, and s the width on the inside. These distances are to be measured along the arcs of circles and not along the chords.

$\frac{1}{2}RS$ = area of circular sector ABO.
$\frac{1}{2}(R - d)s$ = area of circular sector CDO.
$\frac{1}{2}[RS - (R - d)s]$ = area $ACDB$.

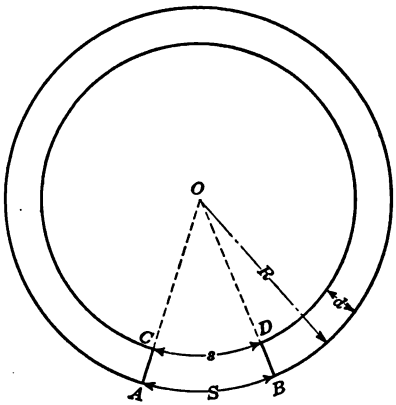


FIG. 42.

Before the last equation can be used we must find an expression for s, for this is, as yet, unknown. From geometry,

$$\frac{s}{S} = \frac{R - d}{R},$$

or

$$s = \frac{S(R - d)}{R},$$

Substituting for s in the above formula for the area of $ACDB$ we have

$$\text{area } ACDB = \tfrac{1}{2}\left[SR - \frac{S(R - d)^2}{R}\right]$$

$$= \frac{S}{2R}[R^2 - (R - d)^2]$$

$$= \frac{S}{2R}(2R - d)(d).$$

[1] This is not the true form of the opening in the silo; the exercise is only introduced here to illustrate how the formula is built up.

Multiplying this area by the height, h, of the opening, and dividing by 27, the volume, v, in cubic yards is

$$v = \frac{Sdh(2R - d)}{54\,R}.$$

ILLUSTRATION: Find v, if $S = 38$ inches (3.1667 feet) and $h = 29.5$ feet, for the cylinder whose volume was computed in exercise 18.

$$\begin{aligned}
\log d &= 9.7659 - 10 \\
\log (2R - d) &= 1.0940 \\
\log S &= 0.5006 \\
\log h &= 1.4698 \\
\log \text{ numerator} &= 2.8303 \\
\log R &= 0.8129 \\
\log 54 &= 1.7324 \\
\log \text{ denominator} &= 2.5453 \\
\log v &= 0.2850 \\
v &= 1.928 \text{ cubic yards.}
\end{aligned}$$

Find v for the set of dimensions given in exercise 19 if $S = 40$ inches, and $h = 21$ feet.

21. Let Fig. 43 represent a vertical cross section of a conical concrete roof of a silo. Find the number of cubic yards of concrete in the roof, assuming no opening. Let k, r, and α be the known dimensions as represented in the figure. The volume sought is the difference between the volume of $AA'C'B'BDA$ and of $ACBDA$. The first-named volume, however, is the sum of the volume of the right circular cone $A'C'B'D'A'$ and the volume of the circular disc $AA'D'B'BDA$; and since the volume of $A'C'B'D'A'$ is equal to the volume of the right circular cone $ACBDA$, the required volume is equal to the volume of the circular disc $AA'D'B'BDA$, or $\pi r^2(BB')$. But, $BB' = \frac{k}{\cos \alpha}$. The final formula for the volume, u, then becomes

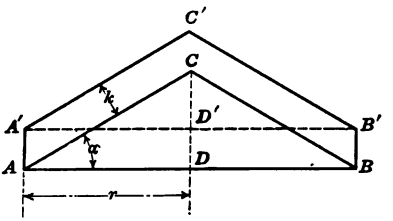


FIG. 43.

$$u = \frac{\pi k r^2}{\cos \alpha}.$$

Find u if $r = 6.25$ feet, $k = 3$ inches, and $\alpha = 30°$.

22. Fig. 44 represents a water tank in the form of a frustum of a right circular cone. Let R and r be, respectively, the radii of the lower and upper bases. Let H be the altitude. Let x be the depth of the water in the tank. AB is a vertical glass tube connected with the tank as shown in the figure. The surface of the water in the tube at C rises to a level with the surface of the water in the tank; and as water is added to or taken from the tank, C will rise or fall.

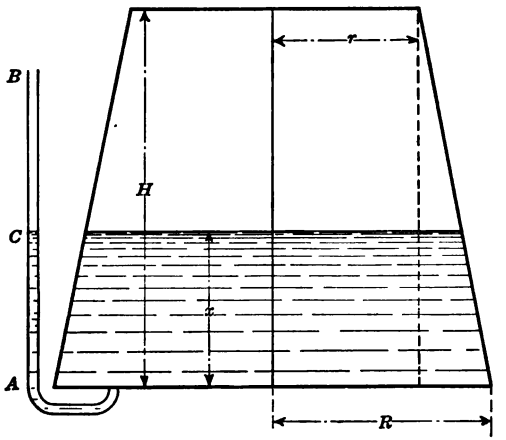


FIG. 44.

If a uniform scale with its zero on a level with the bottom of the tank be placed behind the tube, the reading upon it as indicated by C will give the depth of the water within the tank, and will not give, excepting indirectly, the volume of water within the tank.

The problem is to construct a non-uniform scale, such that, when placed behind the tube, AB, it shall give the direct reading for the volume of water in the tank. It is at once evident that the zero for the scale must be on a level with the bottom of the tank, and that the divisions corresponding to equal increments of volume of water are farther and farther apart as the upper end of the scale is approached. This scale may be constructed by running known quantities of water into the tank and marking the position of the point C; or it may be constructed by computing, from a formula, values of x

for given values of the volume. Let V represent the volume of water in the tank. Show that

$$V = \frac{\pi R^2}{3k} - \frac{\pi k^2}{3}\left[\frac{R}{k} - x\right]^3$$

where $k = (R - r)/H$.

Before solving for x, make the following abbreviations. Let $\frac{\pi R^2}{3k} = \alpha$, $\frac{\pi k^2}{3} = \frac{1}{\beta}$, and $\frac{R}{k} = \gamma$. Then

$$V = \alpha - \frac{1}{\beta}(\gamma - x)^3,$$

or

$$x = \gamma - \sqrt[3]{\beta(\alpha - V)},$$

where α, β, and γ are constants for a given tank. From this formula we may compute x corresponding to any value or set of values for V. These values of x are to be measured off from the zero upon the scale, and the points thus located are to be marked, not with the x values, but with the values given to V.

Compute x for every interval of 20 cubic feet in the value of V ranging from 0 to 600 cubic feet, if $R = 5$ feet, $r = 3$ feet, and $H = 10$ feet. With the values of x computed construct a miniature scale on which 1 inch represents 1 foot. Mark the divisions representing the even hundreds of cubic feet much longer than the subdivisions representing the even twenties. Number only the longer divisions, as 0, 100, 200, 300, . . .

In performing the numerical work, follow the outline given below. Place the values of V, as 20, 40, 60, 80, 100, 120, 140, . . . , 600 upon the first horizontal line of a sheet of paper form *M7*. Subtract these values from the value of α, and place the differences on the second horizontal line. The values of α, γ, and log β should be computed at some out-of-the-way place upon the lower part of the calculation sheet. In subtracting V from α, write α at the lower edge of a slip of paper and by sliding the slip along, the thirty subtractions may be performed with the value of α written but once. Upon the third horizontal line place the logarithms of $\alpha - V$, to which add the logarithm of β and place the sums upon the fourth line. Divide these logarithms by 3, placing the quotients upon the fifth line. Take out the numbers corresponding to the logarithms given on line 5, writing them on line 6. Subtract these numbers from γ, placing the differences on the seventh line, and the numerical work is complete.

In performing the numerical work, several sheets of computation

paper may be used, or by dropping down about ten lines, one sheet may be crossed several times with values of V.

Before constructing the miniature scale, it is advisable first to plot the values for V and x upon a sheet of squared paper. By so doing any appreciable error in an x will be apparent by the corresponding point not falling upon a smooth curve drawn through the plotted points. The value of x for any point not falling upon the curve should be recomputed.

The values for α, β, and $\log \gamma$ should be computed several times to insure their correctness.

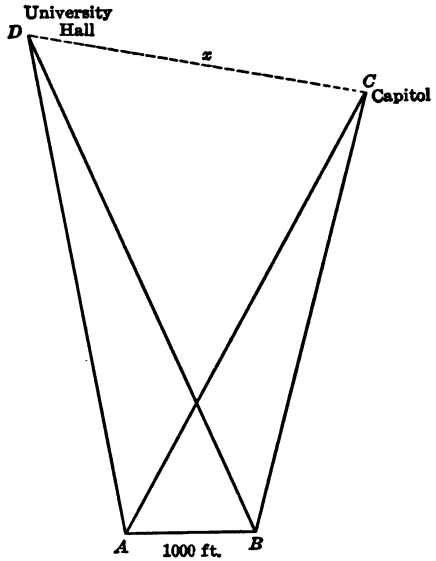


FIG. 45.

23. To find the distance from the flagstaff on University Hall to the flagstaff on the Capitol building, a base line, AB, 1000 feet long, Fig. 45, was laid off on Lakeside Ave., South Madison, and the following angles measured:

$$CBA = 109° 53' 15''.$$
$$DBA = 67° 35' 45''.$$
$$DAB = 105° 1' 30''.$$
$$CAB = 63° 2' 45''.$$

From the triangle ADB compute the length of AD and of DB. From the triangle ACB compute the length of AC and of CB. Compute x from the triangle ACD, and also from the triangle DBC, and take the mean of the two values.

24. To find the height of the ball on the flagstaff on University Hall above the water table of North Hall, the following measurements were taken with the transit placed at the points A and B, 100 feet apart, Fig. 46.

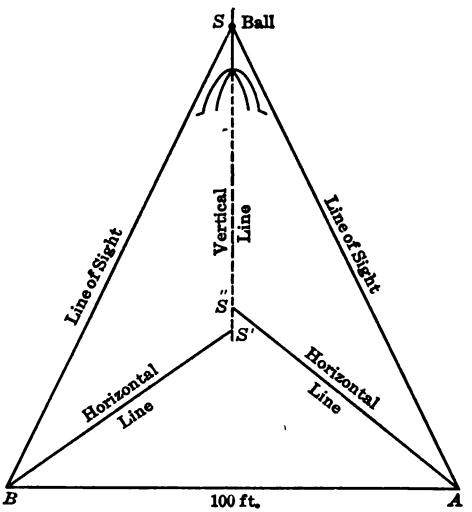


FIG. 46.

The angle of elevation of S at A, $\angle SAS'' = 27° 51' 30''$.
The angle of elevation of S at B, $\angle SBS' = 28° 35' 30''$.
The horizontal angle $S''AB = 75° 31'$.
The horizontal angle $S'BA = 84° 21'$.

At A the transit was 3.33 feet, and at B, 2.71 feet above the water table of North Hall.

Using the law of sines, calculate the logarithm of the length BS', and of AS''. From the triangle $BS'S$ calculate the length of SS', and from the triangle $AS''S$ the length of SS''. To SS' add 2.71, and to SS'' add 3.33. Take the mean of the last two sums as the height of S above the water table.

CHAPTER IV

[THE ELLIPSE]

64. The Ellipse Defined. We shall define the **ellipse** as the curve of intersection of a plane with the surface of a right circular cylinder. This definition includes the circle as a special case of an ellipse.

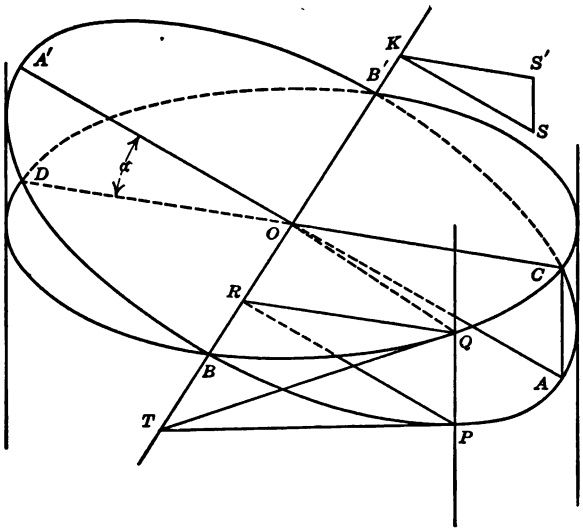


FIG. 47.

In laying out the pattern for the saddle of a circular ventilator, exercise 3, § **42**, an ellipse was constructed by drawing a curve through plotted points. In this exercise the equation of the curve was derived for the case in which the cutting plane, the roof, made an angle of 60° with the axis of the cylinder. The method of finding the equation is, however, the same for all values of the angle.

65. The Equation of the Ellipse. Let $BCB'DB$, Fig. 47, be a normal cross section of a right circular cylinder. The curve $BCB'DB$ is then a circle. Let $BAB'A'B$ be the intersection of the cylinder with a plane inclined to the normal cutting plane at an angle α. The curve $BAB'A'B$ is then, by definition, an ellipse. Let the normal and the inclined cutting planes intersect in the diameter of the cylinder, BB'. Let the radius of the cylinder be b.

In the plane $AB'A'BA$ let O be the origin, OB the positive direction of the X-axis, and OA the positive direction of the Y-axis, of a rectangular coördinate system. If P be any point on the curve $AB'A'BA$, OR is its x-coördinate and RP its y-coördinate.

Let Q be the point of intersection of the normal section with the element passing through P.

Then
$$OR^2 + RQ^2 = OQ^2,$$

or
$$x^2 + RQ^2 = b^2,$$

or, since
$$RQ = RP \cos PRQ = y \cos \alpha,$$
$$x^2 + y^2 \cos^2 \alpha = b^2,$$

or
$$\frac{x^2}{b^2} + \frac{y^2 \cos^2 \alpha}{b^2} = 1,$$

which is the equation of the ellipse.

When α is zero, this equation reduces to $x^2 + y^2 = b^2$, the equation of a circle, the normal cross section of the cylinder. As α increases from 0° to 90°, cos α decreases from 1 to 0, and the coefficient of y^2, $\frac{\cos^2}{b^2}$, decreases from $\frac{1}{b^2}$ to 0. When $\alpha = 90°$, the equation of the ellipse reduces to $\frac{x^2}{b^2} = 1$, or $x^2 = b^2$, or $x^2 - b^2 = 0$, or $(x - b)(x + b) = 0$. The last equation is equivalent to the two equations, $x - b = 0$, and $x + b = 0$, or $x = b$ and $x = -b$; for if the product of two expressions is equal to

zero, the equation is satisfied by equating either factor to zero.[1] Each equation, $x = b$ and $x = -b$, represents a straight line parallel to the Y-axis, b units from it, but upon opposite sides of the origin. This is exactly what should be expected, for when $\alpha = 90°$, the cutting plane becomes parallel to the elements of the cylinder and the curve of intersection reduces to two straight lines.

66. General Shape of the Ellipse. Let OA, Fig. 47, be represented by a. Since $OC = b$, and since $\angle AOC = \alpha$, it follows that $\frac{b}{a} = \cos\alpha$ or $\frac{b}{\cos\alpha} = a$. Substituting this value for $\frac{b}{\cos\alpha}$ in the equation of the ellipse, it reduces[2] to

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1,$$

or, upon solving for y,

$$y = \pm \frac{a}{b}\sqrt{b^2 - x^2}$$

This equation shows that for every value assigned to x there correspond two values for y, numerically equal but with opposite

[1] It will be recalled that the method of replacing one equation by two equations is that which may be used in solving quadratic equations. (See § 15).

Let $x^2 - x - 6 = 0$ be the equation to be solved. By factoring the left-hand side, this equation may be written $(x + 2)(x - 3) = 0$. To solve an equation for x is to find *all* values such that when they are substituted for x in the equation, the equation is satisfied, *i.e.*, the left-hand side reduces identically equal to the right-hand side. To solve then the above equation, considering it written in the second form, means to find *all* values which when substituted for x reduce the left-hand side to zero. To make the left-hand member of this equation zero in all possible ways is to equate each factor to zero. Thus $x + 2 = 0$ and $x - 3 = 0$. But, for $x + 2$ to be equal to zero, x must equal -2, and for $x - 3$ to be equal to zero, x must equal 3. Therefore, -2 and 3 are the only values of x which when substituted for x in the original equation satisfy that equation, *i.e.*, -2 and 3 is the solution of the equation.

In general, if the right-hand member of an equation is zero, and if the left-hand member breaks up into two or more factors, the equation is equivalent to the system of equations formed by equating each factor to zero. Finding all solutions of the system of equations gives all solutions of the original equation.

[2] It must be remembered that Fig. 47 is only a perspective drawing of the cylinder and the two cuttng planes, and that we are at present concerned with the curve $AB'A'BA$ as it appears upon the plane of the paper. We must imagine that we are looking down upon the inclined cutting plane in such a way that OB extends to our right and that OA is the upward vertical direction. The outline would then appear to us as in Fig. 48.

signs. Geometrically this means that the curve is symmetrical with respect to the axis of x. For if the curve were plotted point by point, the two numerically equal values of y, corresponding to a single value of x, show that there are two points upon the curve at the same distance from the X-axis; and the opposite signs of the two values of y show that one point is above the X-axis while the other is below. These two points, since they correspond to a single value of x, are located upon the same side of, and at the same distance from, the Y-axis. The line joining the two points is then perpendicular to, and bisected by, the X-axis. Thus the two points are symmetrical with respect to that axis. Since all points of the curve occur in pairs symmetrically situated with respect to the X-axis, the curve is said to be symmetrical with respect to that axis. As an illustration: let the radius, b, of the cylinder be 2 feet, let the angle between the normal and the inclined cutting planes, α, be 30.°

Then

$$a = \frac{b}{\cos\alpha} = \frac{2}{\cos 30^\circ} = \frac{4}{\sqrt{3}} = \frac{4\sqrt{3}}{3}, \text{ and}$$

$$y = \pm \frac{a}{b}\sqrt{b^2 - x^2} = \pm \frac{2\sqrt{3}}{3}\sqrt{4 - x^2}.$$

When $x = 1$ the last equation gives

$$y = \pm \frac{2\sqrt{3}}{3}\sqrt{4 - 1} = \pm 2.$$

Thus, (1, 2) and (1, − 2) are two points on the curve, each, one unit to the right of the Y-axis. One point is two units above the X-axis, and the other is the same distance below.

From the equation $y = \pm \frac{a}{b}\sqrt{b^2 - x^2}$ it is seen that y increases in numerical value as the quantity under the radical increases, and decreases as the quantity under the radical decreases. The quantity under the radical, $b^2 - x^2$, increases as x decreases in numerical value, and decreases as x increases. The value of $b^2 - x^2$, which is b^2 when x is zero, decreases to zero as x increases to b, and becomes negative when the value of x exceeds that of b.

The numerical value of y, which has a maximum when x equals zero, decreases to zero as x increases to b. When the value of x exceeds b, y becomes imaginary,[1] since the number under the radical sign is then negative. This says, geometrically that for all values of x greater than b, there is no real value of y and no point can be found whose coördinates satisfy the given equation; or the curve does not extend to the right of a vertical line, b units to the right of the Y-axis.

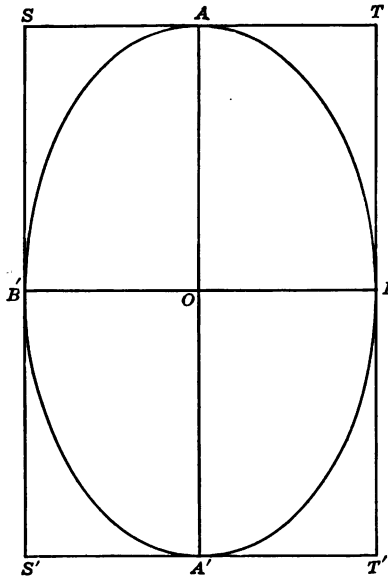


FIG. 48.

The curve is symmetrical with respect to the Y-axis. This fact is shown by its equation, since x occurs only as the square. If two values are assigned to x, numerically equal but with opposite signs, the corresponding pairs of values for y are the same. Which says, geometrically that for any point of the curve there is another point symmetrical to the first point with respect to the Y-axis.[2]

From the preceding discussion it is seen that the curve does not extend to the right of the line $T'T$, Fig. 48, nor to the left of the line SS'; that the curve does not extend above the line ST nor below the line $S'T'$. Therefore the curve is entirely within the rectangle of dimensions $2a$ and $2b$. These conclusions are geometrically apparent from Fig. 47.

[1] The even root, as the square root, the fourth root, the sixth root, etc., of a negative quantity is called imaginary.

[2] That the curve is symmetrical with respect to the Y-axis may also be shown by first solving the equation for x. For, in this form, it is readily seen that, for any value of y, numerically less than a, these are two numerically equal values of x, one positive and one negative.

$OB,(=b)$ and $OA,(=a)$ are called the **semi-axes** of the ellipse.[1]

67. A Geometric Method of Locating Points upon an Ellipse. With O as center, Fig. 49, and with OB, or b, and OA, or a, as radii, draw two circles. From O draw any line making an angle θ with the positive direction of the axis of x, cutting the smaller

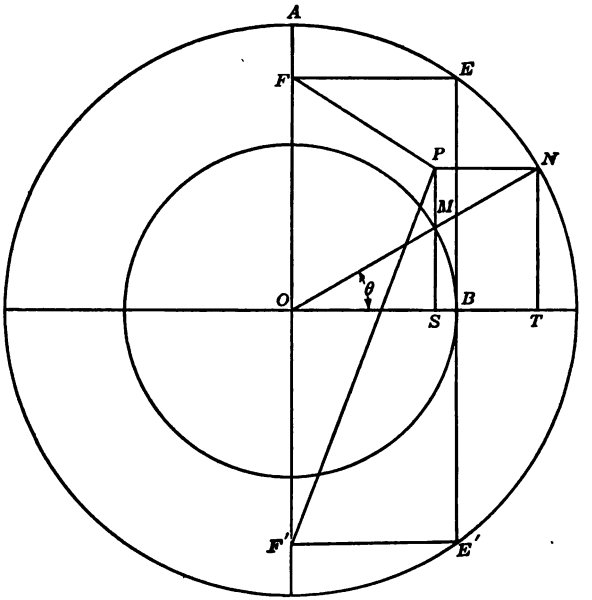


FIG. 49.

circle at the point M and the larger circle at the point N. Through M draw a line parallel to the Y-axis, and through N a line parallel to the X-axis. We shall show that P, their point of intersection, is a point upon the ellipse. The x-coördinate, OS, of the point P

[1] In text-books on Analytic Geometry it is customary to take the major semi-axis of the ellipse coinciding with the axis of x. The ellipse would then appear as Fig. 48 rotated through 90°. Its equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $b > a$.

is $OM \cos\theta$, or $x = b\cos\theta$. The y-coördinate, SP, or TN, of the point P is $ON \sin\theta$, or $y = a\sin\theta$. From these two equations,

$$\frac{x}{b} = \cos\theta$$

and

$$\frac{y}{a} = \sin\theta,$$

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = \cos^2\theta + \sin^2\theta,$$

or, since $\sin^2\theta + \cos^2\theta = 1$,

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1.$$

This is the equation of the ellipse. Therefore, P is a point upon an ellipse. By the above method any number of points may be located upon the ellipse and the curve drawn through them.

68. The Foci; the Eccentricity; the Sum of the Focal Radii. At B, Fig. 49, draw a line tangent to the smaller circle cutting the larger circle at the points E and E'. From E and E' drop perpendiculars upon the Y-axis meeting it at the points F and F'. F and F' are called the **foci** of the ellipse, and the lines FP and $F'P$, lines drawn from the foci to any point P of the ellipse, are called the **focal radii** for the point P.

We shall show that $FP + F'P = 2a$ (a constant for all points upon the ellipse).

Let FP be abbreviated by r_1 and $F'P$ by r_2, and OF (= OF') by c. Then

$$r_1^2 = x^2 + (c - y)^2,$$

or

$$r_1^2 = x^2 + y^2 + c^2 - 2cy,$$

or

$$r_1^2 = b^2\cos^2\theta + a^2\sin^2\theta + c^2 - 2ca\sin\theta.$$

It is easily seen from the rectangle $OBEF$ that $c^2 = a^2 - b^2$, whence,

$$r_1^2 = b^2\cos^2\theta + a^2\sin^2\theta + a^2 - b^2 - 2ac\sin\theta,$$

or since $\cos^2\theta = 1 - \sin^2\theta$,

$$r_1^2 = b^2(1 - \sin^2\theta) + a^2\sin^2\theta + a^2 - b^2 - 2ac\sin\theta,$$

or $$r_1^2 = (a^2 - b^2)\sin^2\theta - 2ac\sin\theta + a^2$$
$$r_1^2 = a^2 - 2ac\sin\theta + c^2\sin^2\theta$$
$$r_1 = a - c\sin\theta.$$

Similarly it can be shown that
$$r_2 = a + c\sin\theta.$$

Upon adding,
$$r_1 + r_2 = 2a,$$
or the sum of the lengths of the two focal radii is a constant.

This property of the ellipse furnishes an easy method for its construction. From the relation $c^2 = a^2 - b^2$, locate the foci, F and F'. Fasten the ends of a string, whose length is $2a$, at the points F and F'. Place a pencil point in the loop of this string and draw taut. By moving the pencil, allowing the string to slip around it, an ellipse will be described.

We shall prove the converse of the above theorem: The locus of a point, moving in a plane such that the sum of its distances from two fixed points remains constant, is an ellipse.

Let F and F' be the fixed points, and P any point upon the curve. Let the Y-axis pass through F and F'. Let O, the midpoint of FF', be the origin. Let the distance OF $(= OF')$ be represented by c. Let FP be represented by r_1 and $F'P$ by r_2.

$$r_1^2 = x^2 + (c - y)^2 \tag{1}$$
$$r_2^2 = x^2 + (c + y)^2. \tag{2}$$

By adding,
$$r_1^2 + r_2^2 = 2(x^2 + y^2 + c^2). \tag{3}$$

By subtracting,
$$r_2^2 - r_1^2 = 4cy. \tag{4}$$
$$r_1 + r_2 = 2a \text{ (constant sum) by hypothesis.} \tag{5}$$

Dividing (4) by (5),
$$r_2 - r_1 = \frac{2cy}{a}. \tag{6}$$

By adding (5) and (6),
$$r_2 = a + \frac{cy}{a}. \tag{7}$$

By subtracting (6) from (5),
$$r_1 = a - \frac{cy}{a}. \tag{8}$$

Squaring (7) and (8) and adding,

$$r_1^2 + r_2^2 = 2(a^2 + c^2y^2/a^2). \tag{9}$$

Substituting in the first member of (3) and canceling the common factor 2,

$$a^2 + \frac{c^2y^2}{a^2} = x^2 + y^2 + c^2.$$

Clearing of fractions,

$$a^4 + c^2y^2 = a^2x^2 + a^2y^2 + a^2c^2,$$

or

$$a^2x^2 + (a^2 - c^2)y^2 = a^2(a^2 - c^2). \tag{10}$$

Dividing (10) by the quantity $a^2(a^2 - c^2)$,

$$\frac{x^2}{a^2 - c^2} + \frac{y^2}{a^2} = 1.$$

But, this equation is of the form

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1.$$

which we have shown to be the equation of the ellipse.

The distance $OF = OF' = c$ is called the **focal distance.** The ratio, c/a, which is never greater than unity, is called the **eccentricity** of the ellipse. Referring to Fig. 47 the focal distance is equal to AC, and the eccentricity is the sin α. A circle then is an ellipse with zero eccentricity.

69. The Area of the Ellipse. Let $ABCD$ and $ABC'D'$, Fig. 50, be two planes inclined to each other at an angle α. Let $EFG'H'$ be a rectangle in the plane $ABC'D'$ with its base EF in (or parallel to) the line of intersection of the two planes. Let $EFGH$ be the orthographic projection of $EFG'H'$ upon the plane $ABCD$.[1] Since $EH = EH' \cos \alpha$, the area of $EFGH$ = (area of $EFG'H'$) cos α.

If, instead of a rectangle, an irregular area, as $E'F'G'H'$,

[1] If from a point a perpendicular is let fall upon a plane, the foot of the perpendicular is called the orthographic projection of the point upon the plane. If from every point of a curve a perpendicular is let fall upon a plane, the locus of the foot of this perpendicular upon the plane is called the orthographic projection of the curve upon the plane. If the bounding curve or curves of any surface be projected orthographically upon a plane, the area bounded by the projection is called the orthographic projection of the area of the given surface upon the plane.

Fig. 51, on the plane $ABC'D'$ be projected orthographically upon the plane $ABCD$, divide the projected area into strips by equidis-

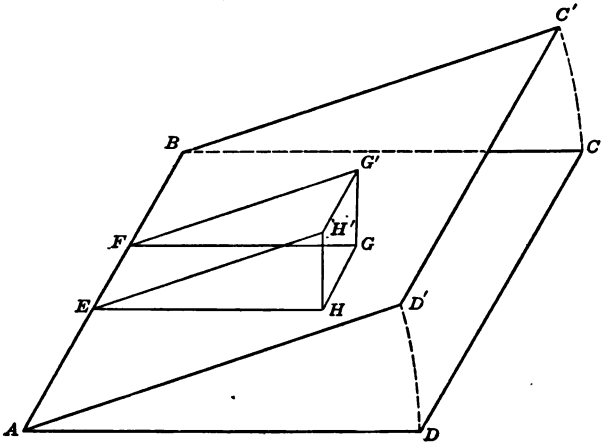


Fig. 50.

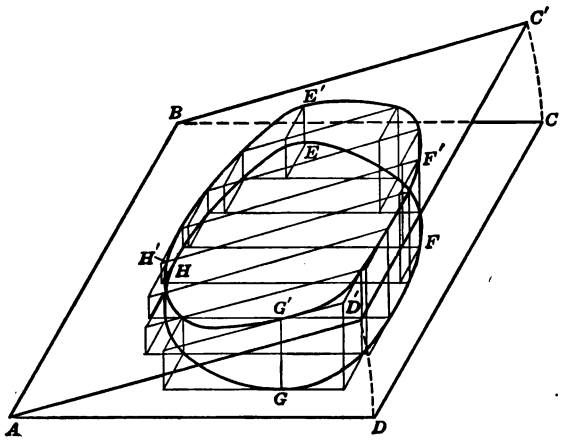


Fig. 51.

tant parallel lines drawn perpendicular to AB, and from the points of intersection of each of these lines with the boundary of

the area draw lines to the right until they meet the next parallel. In this way, on the plane $ABC'D'$, is formed a series of rectangles which project into another series of rectangles on the plane $ABCD$. From what was shown above it is seen that the sum of the areas of the projections of these rectangles upon the plane $ABCD$ is equal to the sum of the areas of the projected rectangles multiplied by the cos DAD'.

As the distance between the parallel lines is made smaller and smaller, *i.e.*, as the number of strips is increased indefinitely, the sum of the areas of the rectangles in the plane $ABC'D'$ approaches nearer and nearer the irregular area $E'F'G'H'$; and the sum of the areas of the rectangles in the plane $ABCD$ approaches nearer and nearer the area $EFGH$, the projection of $E'F'G'H'$. Therefore, area $EFGH$ = (area $E'F'G'H'$) cos DAD'. Then (Fig. 47) the area of the circle $BCB'D$ = (area of the ellipse $ABDB'$) cos α, or, since cos $\alpha = b/a$, and since area of the circle $= \pi b^2$,

$$\pi b^2 = (\text{area of ellipse})\frac{b}{a},$$

or

$$\textbf{area of ellipse} = \pi \textbf{ba}$$

70. The Orthographic Projection of a Circle. Let $ABCD$, Fig. 52*a*, be a circle with radius a. Let AOC, Fig. 52*b*, be an end elevation of the same circle. Rotate this circle about BOD as an axis through an angle, β, to the position $A''OC''$. Project the rotated circle upon its original plane, into the curve $A'BC'D$. We shall show that $A'BC'D$ is an ellipse. Take any point P upon the original circle. It rotates into the point P'', and P'' projects into P'. The equation of the circle is $x^2 + y^2 = a^2$, where $x = MP$. To get the equation for the curve $A'BC'D$ replace MP by its equal $MP'/\cos\beta$. (See Fig. 52*b*.) Whence,

$$\frac{MP'^2}{\cos^2\beta} + y^2 = a^2.$$

Since MP' is the x-coördinate of P',

$$\frac{x^2}{\cos^2\beta} + y^2 = a^2,$$

or

$$\frac{x^2}{a^2 \cos^2 \beta} + \frac{y^2}{a^2} = 1.$$

Replacing $a \cos \beta$ by $b(= OC'')$,

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1,$$

the equation of an ellipse. The focal distance, $\sqrt{a^2 - b^2}$, is equal to $C'C''$, and the eccentricity $c/a = \sin \beta$.[1]

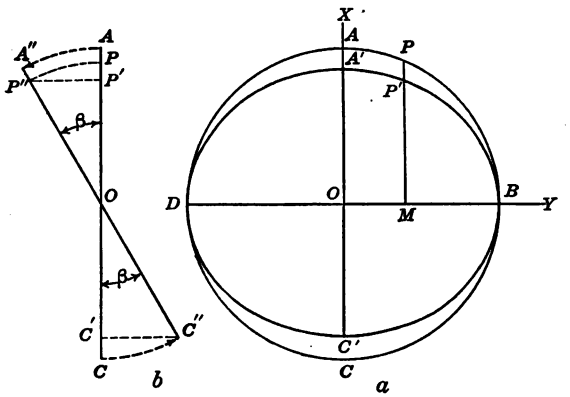


FIG. 52.

[71.] Tangent to an Ellipse. Let PTQ, Fig. 47, be a tangent plane to the cylinder with PQ the line of tangency. Let QT and PT be, respectively, the lines of intersection of the tangent plane with the horizontal and the inclined cutting planes. TQ is tangent to the circle $DBQCB'$ at the point Q, and TP is tangent to the ellipse $BPAB'A'$ at the point P. These tangent lines meet at the point T upon the shorter axis of the ellipse produced. This fact will be made use of in drawing the tangent to an ellipse.

Let $ABA'B'$, Fig. 53, be an ellipse to which a tangent line is to be drawn at the point P. Draw the circle with $B'B$ as diameter. From P drop a perpendicular upon $B'B$ cutting the circle at Q.

[1] It can be shown that the section of a right circular cone made by a plane cutting all of its elements on the same side of the vertex is an ellipse.

Through Q draw QT tangent to the circle (perpendicular to OQ). Draw TP, which will be tangent to the ellipse at the point P.

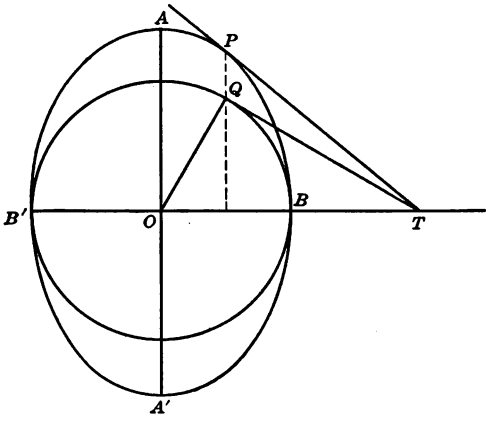


FIG. 53.

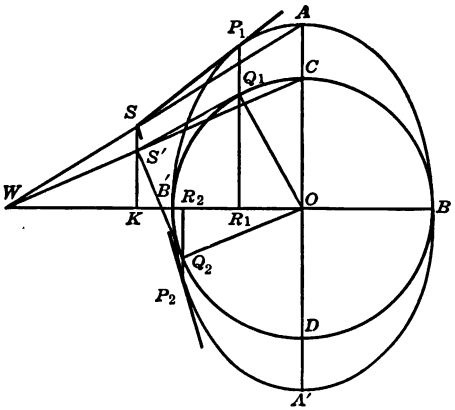


FIG. 54.

It is left for the student to show that the above statement is correct.

We shall now show how to draw a tangent to an ellipse through

a point S, not upon the curve. Let S, Fig. 47, be a point in the plane of the ellipse, through which the tangent line is to be drawn. Draw SS' parallel to PQ, intersecting the plane OCQ at S'. Draw SK and $S'K$ perpendicular to BB' From the similar right triangles $SS'K$ and ACO it follows that:

$$S'K/SK = CO/AO$$

Let $ABA'B'$, Fig. 54, be the ellipse to which the tangent is to be drawn from the point S. Draw ASW. Draw CW. Draw SK perpendicular to BOW, cutting CW at the point S'. From S' draw the two tangents to the circle $BCB'D$. Let Q_1 and Q_2 be the points of tangency. Draw $R_1Q_1P_1$ and $R_2Q_2P_2$ perpendicular to $B'OB$ cutting the ellipse in the points P_1 and P_2. Draw P_1S and P_2S. P_1S and P_2S are the tangents to the ellipse drawn from the points S. P_1S and Q_1S' intersect upon $B'BW$; and P_2S and Q_2S' intersect upon $B'BW$. It is left for the student to show that the above statements are correct.

Exercises

1. Find the value of each semi-axis, the focal distance, and the eccentricity for each of the following:

(*a*) $\frac{x^2}{4} + \frac{y^2}{9} = 1.$ (*c*) $100x^2 + 4y^2 = 400.$

(*b*) $\frac{x^2}{16} + \frac{y^2}{25} = 1.$ (*d*) $7x^2 + 3y^2 = 11.$

2. Construct an ellipse by the method given in § **67** if $a = 5$ inches, and $b = 4$ inches.

3. Construct an ellipse by the method given in § **68** if $a = 4$ and $\alpha = 30°$.

4. Find the weight of metal removed in cutting the opening in the saddle for a circular ventilator $3\frac{1}{2}$ feet in diameter, if the rafters make an angle of 30° with the horizontal, and if the ventilator is made of 20 gauge iron. Twenty gauge iron is 0.038 inch thick, U. S. standard. The specific gravity of steel is 7.83. A cubic foot of water weighs 62.5 pounds.

[5.] Show that a right elliptical cylinder has a circular cross section. If,

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

is the equation of its normal cross section, find the angle between the axis of the cylinder and the plane giving the circular section.

[6.] Show that every section of a right elliptical cylinder is an ellipse.

[7.] A circular window in the east wall of a building is 4.6 feet in diameter. The wall runs true north and south. The rays of light from the sun pass through the window and fall upon a horizontal floor. Find the area of the floor in the sunlight if the angle of elevation of the sun is 57° 42′.6, and if it is 37° 27′.7 east of the meridian.

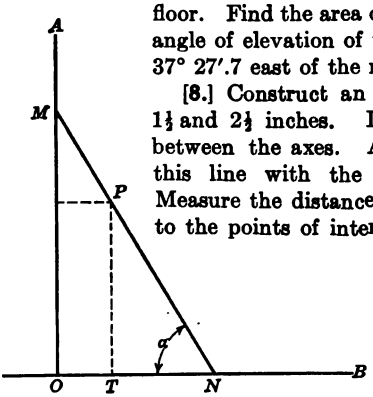


FIG. 55.

[8.] Construct an ellipse with semi-axes equal to $1\frac{1}{2}$ and $2\frac{1}{2}$ inches. Draw a line bisecting the angle between the axes. At the point of intersection of this line with the ellipse, draw a tangent line. Measure the distances from the center of the ellipse to the points of intersection of the tangent line with the axes of the ellipse produced.

[9.] Construct an ellipse with semi-axes equal to $2\frac{1}{2}$ and $3\frac{1}{2}$ inches. Through its center draw a line making an angle of 30° with the minor axis. From a point of this line 2 inches from the center draw two tangents to the ellipse.

[10.] Through the point of tangency of the tangent line drawn in exercise 8 draw a line perpendicular to the tangent line. With the protractor measure the angles between this perpendicular and the focal radii.

[11.] Let OA, Fig. 55, be the vertical wall of a building; OB a horizontal pavement; MN a ladder of length $a + b$. P is a point of the ladder such that $MP = b$ and $NP = a$. Show that as the foot of the ladder is drawn out along the pavement, P moves along the arc of an ellipse.

HINT: Consider OB as the X-axis and OA the Y-axis of a rectangular coördinate system. OT ($= x$) and TP ($= y$) are the coördinates of P. $\frac{y}{a} = \sin \alpha$, and $\frac{x}{b} = \cos \alpha$. Square these equations and add.

CHAPTER V

[THE SLIDE RULE]

72. Accuracy of Logarithmic Computation. The logarithms given in tables are only approximations. In a four-place table, they are correct to the nearest fourth decimal place; in a five-place table to the nearest fifth decimal place; in a six-place table to the nearest sixth decimal place; etc.

The $\log \pi$	= 0.497	to three places,
	0.4971	to four places,
	0.49715	to five places,
	0.497150	to six places,
	0.4971499	to seven places, and
	0.49714987	to eight places.

The $\log \pi$ is given correct to three, four, five, six, seven and eight places of decimals; but none of these values is absolutely correct.

Since the logarithms of numbers taken from tables and used in logarithmic calculation are not absolutely correct, the results of the calculations are not absolutely correct. But in nearly all numerical calculation, results are required accurate only to a certain number of places; and the number of accurate places required is the fact that determines the table to be used. If a set of logarithms are added (corresponding to multiplication) or added algebraically (corresponding to multiplication and division) the errors in the logarithms will be, in most cases, compensating in nature. That is to say, the positive errors and negative errors tend to balance, producing only a small error in the sum.

Below are written ten numbers selected at random. In the second column are given their logarithms to six places; in the third column their logarithms to four places; and in the fourth column the errors in the four-place logarithms as compared with the six-place logarithms.

Number	Logarithm	Logarithm	Error
3.72	0.570543	0.5705	−0.000043
9.81	0.991669	0.9917	+0.000031
7.15	0.854306	0.8543	−0.000006
1.23	0.089905	0.0899	−0.000005
5.06	0.704151	0.7042	+0.000049
7.17	0.855519	0.8555	−0.000019
8.23	0.915400	0.9154	0.000000
3.56	0.551450	0.5514	−0.000050
1.17	0.068186	0.0682	+0.000014
8.75	0.942008	0.9420	−0.000008
Sum.....	.....	6.4431	−0.000037

The sum of the errors is − 0.00004, which represents approximately the error in the sum of the ten logarithms. The mantissa 4431 is found to correspond to the number 2774. Since the tabular difference at this point of the table is 15, the above error, 0.00004 makes only a difference of between 2 and 3 in the next (fifth) place in the number. Thus by using a four-place table in multiplying these ten numbers together, the product is found correct to the nearest fourth place.

If the sum of the ten logarithms had been 6.9872, the tabular difference at this point of the table would have been 5, the error of − 0.00004 would have produced an error in the corresponding number of about 7 in the fifth place. Thus, the result of the product of the ten numbers would have been one unit too small in the fourth place.

We see from this illustration that an error in the logarithm, the sum, may effect the fourth digit one, or possibly two points, if the logarithm stands near the end of the table; while the fourth digit is given correctly if the logarithm stands nearer the beginning of the table.

Multiplying and dividing with a four-place table gives results correct to four places, or correct to three places while the fourth place is in error one or two points only. The error in the product or quotient will usually be less than 2/100 of 1 percent, and in the majority of cases it will be very much less than this.

This is what is meant when we say a four-place table computes accurately to four places.

In taking out the number corresponding to a logarithm, do not, in general, write more than four significant digits if a four-place table is used. If there are more than four digits preceding the decimal point, determine the first four, and suffix ciphers between the fourth digit and the decimal point. Thus, if $\log x = 6.1738$, $x = 1{,}493{,}000$. The fourth digit is determined by interpolation, and three ciphers are suffixed. This number, however, occurring near the beginning of the table, could have been corrected to another place, in which case the fifth digit would have been correct to within three or four points.

In numerical work, *use the smallest table which will insure the required degree of accuracy in the result.* If a five-place table will suffice, *do not* use a six-place table. If a four-place table will suffice, *do not* use a five-place table. If the numerical work of a problem is to be done with four-place logarithms, the student who has had but little practice in logarithmic computation is very apt to believe that by using a five-place table and discarding the last digit he will save the time and trouble required in applying corrections. This method, however, *does not gain the end sought;* for, by using a well-arranged four-place table, corrections can be applied, after a very little practice, more expeditiously and easily than the logarithms can be found from the larger table containing ten times as many numbers.

Another oversight on the part of a large number of students, is the attempt to compute more accurately than the observed data warrant.

Thus, to illustrate, suppose a and b are two readings. They are not absolutely correct. The first has a possible error ϵ_1 and the second a possible error ϵ_2. The two readings are to be multiplied together. What is the possible error of the product? How many places in the product are reliable?

$$(a \pm \epsilon_1)(b \pm \epsilon_2) = ab \pm a\epsilon_2 \pm b\epsilon_1 + \epsilon_1\epsilon_2.$$

The sum of the last three terms in this product, taken with the upper signs, is the largest possible error in the product due to the errors ϵ_1 and ϵ_2 in the factors a and b, respectively. $\epsilon_1\epsilon_2$ is

small compared with $a\epsilon_2 + b\epsilon_1$, and may be neglected. The possible error in the product is then $\pm (a\epsilon_2 + b\epsilon_1)$.

Suppose 325 and 521 are the readings. Suppose that in each case 1 in the third place is the least count of the instrument, *i.e.*, there is a possible error of 0.5 in each reading. Then,

$$\begin{aligned} a &= 325 \\ b &= 521 \\ \epsilon_1 &= \epsilon_2 = 0.5 \\ ab &= 169{,}325 \\ a\epsilon_1 + b\epsilon_2 &= 423 \end{aligned}$$

Thus we see that the product of the two numbers whose measurements were attempted lies between 169,325 + 423 and 169,325 − 423. This shows that the product may be correct to three places only, and should be written 169,000. The digits following the 9 in the product *ab* have no significance since all may be incorrect; hence the digits should not be written, but ciphers substituted in their places.

In the above illustration, even a four-place table carries the accuracy of the work beyond what is required. In this problem the calculation could have been done with a three-place logarithmic table, or with a *slide rule.*

73. The Principle of the Slide Rule. Let *C* and *D*, Fig. 56, represent two uniform scales. The two scales are so placed that the zero mark of *C* stands opposite the 3 mark of *D*. Any number on *D* to the right of 3 is the sum of the number directly above it on *C*, and 3. Thus, 5(read on scale *D*) is the sum of 2 and 3; 6 is the sum of 3 and 3; $4\frac{1}{2}$ is the sum of $1\frac{1}{2}$ and 3; etc. In this way any number may be added mechanically to the number 3. By placing the 0 of *C* above any point of *D*, any other number may be added mechanically to the number corresponding to this point of the *D* scale.

It is immaterial whether the 0 of *C* or the 0 of *D* is set to correspond to one of the numbers to be added. If the 0 of *D* is set, the sum is read from the *C* scale.

By a similar process, subtraction may be performed mechanically. Thus, if 3 is to be subtracted from 6, move scale *C* so that

3 upon it stands above 6 of D, the number of D below the 0 of C, or 3, is the difference.

In exercise 10, § **41**, there were constructed two logarithmic scales, each in juxtaposition with a uniform scale.[1] These two pairs of scales are represented (reduced) in Fig. 57. Bend scales C and D along the horizontal lines; turn the lower part, the uniform scale, of C back behind the non-uniform scale; and turn the upper part of D back behind the lower part. Now place these two scales together as shown in Fig. 58. The lines marked 1 and 10 of these scales are called indices.

If we bring the left-hand index of C opposite 2 of D, any number to the right of 2 on the D scale will be the product of the corresponding number of C, and 2. Thus, in the figure, 6 of D stands beneath 3 of C; 6 is the product of 3 and 2. To multiply 2 by 3 mechanically, bring the index of the upper scale over 2 of the lower scale, and read the number on the lower scale which stands directly under 3 of the upper scale. This number is the product. For, upon the back of each scale, C and D is a uniform scale, and by sliding C and D upon each other we are adding mechanically, as was shown above, the numbers of these uniform scales. Thus, on the back of D is a number, which is the sum of the number back of 2 on D and the number back of 3 on C. But, the number back of 2 on D is the logarithm of 2, and the number back of 3 on C is the logarithm of 3; then on the back of D opposite 3 on C is the logarithm of 2 plus the logarithm of 3, or the logarithm of 6, since the sum of the logarithms of two numbers is the logarithm of their product. On the face of D are numbers whose logarithms are on the back. Therefore, opposite the logarithm of 6 upon the back of D, *i.e.*, opposite 3 of C, we have 6, or the product of 2 and 3.

In constructing scales C and D, one unit of length only, of the uniform scale was used. By extending these double scales indefinitely in both directions, the uniform parts will include all algebraic numbers, *i.e.*, all numbers from $-\infty$ to $+\infty$; the non-uniform parts will include all positive arithmetical numbers, *i.e.*, all numbers from 0 to $+\infty$. A portion of one of these scales (the C

[1] If this exercise was not performed when the work on logarithms was taken up, it should be performed now, or at least the description read carefully.

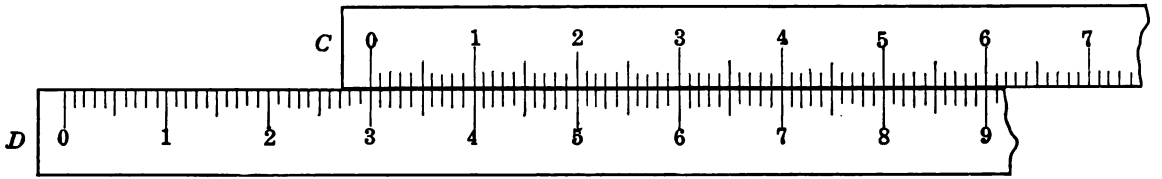


FIG. 56.

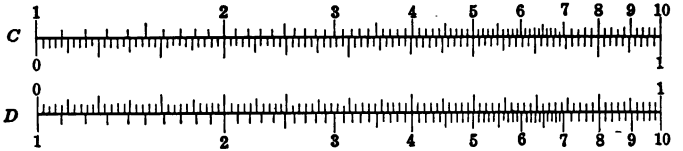


FIG. 57.—Two logarithmic scales each in juxtaposition with a uniform scale.

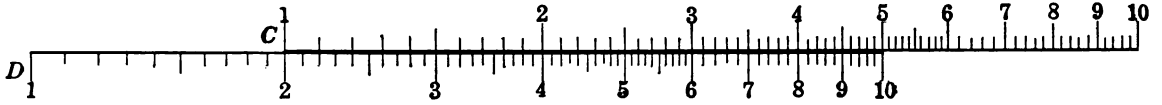


FIG. 58.
Two logarithmic scales in juxtaposition.

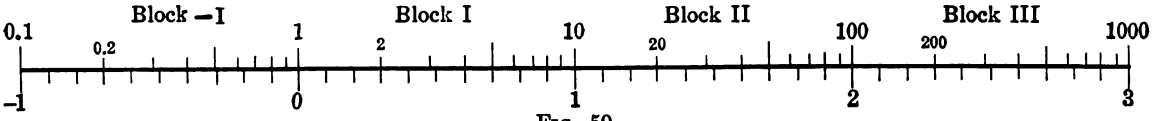


FIG. 59.

scale) is represented in Fig. 59. The extended D scale is the same, but inverted. For convenience of description, different segments of these scales will be referred to as "block I," "block II," etc. By block II is meant that portion of the double scale between 1 and 2 of the uniform part, or between 10 and 100 of the non-uniform part; by block $-$ I is meant that portion of the uniform part between -1 and 0, or between 0.1 and 1 of the non-uniform part, etc.

Following directly from the facts that the mantissas of the common logarithms are independent of the position of the decimal point in the number, and that by moving the decimal point one place to the right or left increases or decreases, respectively, the characteristic of the logarithm by 1, it is seen that all blocks of the double scale, Fig. 59, are identical. To illustrate: the line on the non-uniform part representing 20 is in the same position relative to 10 as the line representing 2 is to 1. This must be so, for the logarithm of 20 equals the logarithm of 10 plus the logarithm of 2; and, since the logarithms are represented on the uniform part, the logarithm of 20 is as far to the right of 1 on the uniform part as the logarithm of 2 is to the right of 0. Since all blocks are the same, one block will suffice for the entire indefinite scale. In using scales C and D, Fig. 58, for mechanical multiplication the operator may disregard the positions of the decimal points in the factors, and point off the proper number of places in the product after the sequence of digits has been found.

Since the index at the right-hand end of C may be looked upon as 1 as well as 10, the right-hand index as well as the left-hand index may be set above one of the factors of the product. It will be seen, however, that in multiplying two numbers together, using only one block each of the C and D scales, only one of the two indices of C can be set over the mutiplicand, for if the other index were set it would throw the product off the D scale into the next block. By interchanging the multiplicand and multiplier the other index of C may be set. To illustrate: let 3 be multiplied by 6. Set the left-hand index of C over 3, then 6 of C falls beyond the D scale. In this case the product may be found either by setting the right-hand index of C over 3 and reading

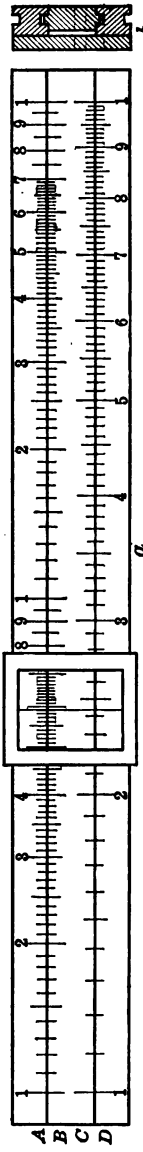


FIG. 60.—The slide rule.

under 6, or by setting the right-hand index of C over 6 and reading under 3.

By means of scales C and D any two numbers may be multiplied together mechanically. The principle involved, *i.e.*, of adding logarithms mechanically, is the underlying principle of the construction of the (Mannheim) slide rule.[1] The operations performed by the mechanism may be compared with logarithmic work as follows. Locating the numbers on the scales may be compared with taking out the logarithms of the numbers and the anti-logarithm of their product, while sliding the C scale above the D scale may be compared to the addition of the logarithms.

Since the logarithms of the factors and the logarithm of their product are not read they need not be printed upon the back of the scales C and D.

74. The Slide Rule. The slide rule consists of three parts: the body, the slide, and the indicator (Fig. 60). The body of the rule consists of a wooden base about 10 inches long,[2] about 1 inch wide, and about 1/8 inch thick, upon one side of which are fastened two parallel strips, each carrying

[1] Many mechanical devices have been constructed for performing special numerical operations. Often they are in the form of one scale sliding over another, and for this reason are also termed, with a proper qualifying adjective, slide rules; as, stadia slide rule, sewer slide rule, horse-power slide rule, etc.

The slide rule is also constructed by placing the logarithmic scales upon two concentric circular arcs, one rotating within the other.

[2] Slide rules are made in three sizes: viz., the 5-inch, about 5 inches long; the 10-inch, about 10 inches long; and the 20-inch, about 20 inches long. The 10-inch is the size commonly used. The 5-inch is less accurate, but is very convenient for carrying in the pocket and for rough estimates.

a logarithmic scale engraved upon celluloid. These two scales are, in Fig. 60*a*, marked *A* and *D*. A transverse section of the rule is shown in Fig. 60*b*. The two upper parallel strips of the body are separated about 1/2 inch, and between them there is placed the slide, a thin strip of wood carrying two logarithmic scales, *B* and *C*. The scales *C* and *D* are the same as those explained in the preceding section, and by means of them, multiplications may be performed.

The indicator *I*, Fig. 60*a*, consists of a piece of glass, about an inch square, mounted in an aluminum frame, which slides in grooves lengthwise the rule. On the under side of the glass, very nearly in contact with the surfaces of the four scales, is etched a fine vertical line, or, "cross line" as it will be called.

Scales *A* and *B* are logarithmic scales *exactly one-half* as long as scales *C* and *D*, *i.e.*, upon *A* and *B* are two blocks of the indefinite logarithmic scale. Scales *A* and *B*, then, have each three indices, one at each end and one at the mid-point. When we come to the explanation of the use of the *A* and *B* scales we shall see that the end indices are considered to correspond to the numbers . . . 0.0001, 0.01, 1, 100, 10,000, . . . , while the central index corresponds to the numbers . . . 0.001, 0.1, 10, 1000, 100,000. . .

The scales *A* and *B* are placed upon the body of the rule so that the left-hand index of *A* is directly over the left-hand index of *D*, and so that the right-hand index of *A* is directly over the right-hand index of *D*. When the cross line of the indicator is set over the left-hand index of *A* it will also be over the left-hand index of *D*; when set over the right-hand index of *A* it will also be over the right-hand index of *D*.

75. The multiplication of two or more numbers will be illustrated by an example. To multiply (37.5) (212) (11.2) (0.315) proceed as follows: Set left-hand index of *C* over 375 of *D*. Set cross line of indicator over 212 of *C*. (Directly under the cross line on the *D* scale is the product of the first two factors, but do not read it.) Set the left-hand index of *C* under cross line (*i.e.*, over the product of the first two factors on *D*). Set cross line over 112 on *C*. Bring the right-hand index of *C* under the cross line. Move cross line over 315 of *C*, and under it on the *D* scale read

the product of the four factors. The digits read are 2805. To locate the decimal point, find mentally a very rough estimate of the product; thus, the product of the first two factors is about ½ of 21,000, or about 7000; the next multiplier (about 10) makes the product of the first three factors about 70,000; the next multiplier is about ½, which makes the product of four factors about 23,000. This shows that there are five digits before the decimal point. The product is then 28,050.

It is to be noticed that in the above operations no readings are taken from the rule excepting the final product.

In a similar way any number of factors may be multiplied together.

If several numbers are to be multiplied by the same number, set the index of *C* over the common factor on *D*. Slide in turn the cross line over the given numbers on *C*, and read off the various products from *D*. In this way all the products can be read with one, or at most two, settings of the slide.

76. Division. The division of one number by another may be performed upon a slide rule in two ways. The principle of one method is that subtracting the logarithm of the divisor from the logarithm of the dividend gives the logarithm of the quotient. The principle of the other method is that multiplying the quotient by the divisor gives the dividend. Both methods will be illustrated by example. To divide 372 by 176, set the slide so that 176 of *C* is directly above 372 of *D*. (To do this easily, first set the cross line over 372 of *D* and then bring 176 of *C* under it.) Read the quotient from *D*, the number under the index of *C*. The quotient is 2.113. By inspection it is seen that the decimal point follows the 2.

This method of division corresponds to the subtraction of logarithms. For, the distance from the index to 372 (measured upon the uniform scale on the back of *D*) is the logarithm of 372. The distance from the index of *C* to the number 176 is the logarithm of 176. Then the distance from the index of *D* to the index of *C* is the difference of these two logarithms, or the logarithm of the quotient; and the number on the face of *D* is then the quotient.

To divide by the second method, set the index of *C* over the divisor, 176, on *D*. Set the cross line over the dividend, 372, on

D. Under the cross line on the C scale read the quotient, 2.113 It is easily seen that this method gives the quotient, for it is the reverse of that used in multiplication.[1]

The second method of division is useful if several numbers are to be divided by a common number; for example, to find the percentages which a set of numbers are of a given number, place the index of C over the common divisor on D, bring the cross line over the dividend on D and read the quotient from C under the cross line. In this way a series of quotients may be found with one, or at most two, settings of the slide.

77. The A and B Scales. Since the "block" on A is just one-half as long as the "block" on D, the numbers (in imagination) on the back of A are just double those directly opposite on the back of D. The numbers on the face of A are then the squares of the numbers directly opposite on the face of D. This follows from the fact that the logarithm of the square of a number is two times the logarithm of the number.

ILUSTRATIONS: To find the square of 3.77, either multiply 3.77 by 3.77 or set the cross line over 3.77 on the D scale and read its square, 14.2, from the A scale. To extract the square root of 7.13, set the cross line over 713 of the left-hand block of the A scale, read off its square root, 2.67, from the D scale. To extract the square root of 71.3, set the cross line over 713 of the right-hand block of A, read off its square root, 8.44, from the D scale.

In extracting square roots care must be exercised to enter the number in the proper block of the A scale.

Scales A and B may be used for multiplication and division in the same way that scales C and D are used. But on account of the A and B scales being only half as long, the results are not as accurate as when C and D are used.[2]

78. The Cube Root.[3] The method of finding cube roots of numbers will be illustrated by examples.

[1] The first method may also be looked upon as the reverse of multiplication.

[2] *The Polyphase Slide Rule* is one having two additional scales. One of these scales gives cubes (and cube roots) directly; the other gives reciprocals of numbers directly.

[3] Some rules are provided with a cube scale by means of which cubes and cube roots may be obtained directly.

Example 1: *Find the cube root of* 8. Set the cross line over 8 on the *A* scale (block I). Move the slide so that the number on the *B* scale under the cross line is the same as the number on the *D* scale under the left-hand index of the *C* scale. This is found to be 2, the cube root of 8.

Example 2: *Find the cube root of* 80. Set the cross line over 8 on the *A* scale (block I). Move the slide so that the number on the *B* scale under the cross line is the same as the number on the *D* scale under the right-hand index of the *C* scale. This is found to be 4.31, the cube root of 80.

Example 3: *Find the cube root of* 800. Set the cross line over 8 on the *A* scale (block II). Move the slide so that the number on the *B* scale under the cross line is the same as the number on the *D* scale under the right-hand index of the *C* scale. This is found to be 9.28, the cube root of 800.

The student will explain why these operations give the cube roots.

79. Other Operations with the Slide Rule. It was the intention to give above only the underlying principle of the slide rule; and to explain its use in connection with the simple operations of multiplication, division, extracting square root, and squaring. It is believed that the student will, after some practice with the slide rule, devise methods of procedure which will involve the fewest number of settings of the slide, and the fewest number of intermediate readings. Pamphlets and books may be procured giving in considerable detail the methods of performing numerical calculation with the slide rule.

In performing numerical calculations with the slide rule, the work should be blocked out as when calculating with logarithms.[1]

Exercises

The student will practice on simple multiplications, divisions, squares, square roots, and cube roots.

[1] Paper, form *M*8, is ruled especially for slide rule calculations.

CHAPTER VI

STATICS

80. Vector and Scalar Quantities. In Physics and in Mechanics we have to deal with two kinds of quantities: vector quantities and scalar quantities.

Quantities, such as velocity, force, acceleration, etc., which require for their complete definition direction as well as magnitude, are called **vector quantities.** When we say a body has a uniform velocity of 10 feet per second, we mean that in every second of time the body moves a distance of 10 feet and in some given direction. Velocity, then, is a vector quantity.

Quantities, such as mass, time, area, volume, etc., having only magnitude, are called **scalar quantities.** In order to convert a gram of ice at zero degrees centigrade into water at the same temperature, 79 units of heat must be added to the mass. This quantity, 79 calories, is a scalar quantity.

81. Graphic Representation of Vector Quantities. In order to represent a vector quantity graphically, choose some unit of length, as the inch, the half-inch, the centimeter, to represent the unit of magnitude; with this unit length, draw a line whose length measures the magnitude of the quantity, and whose direction is that of the vector quantity. To illustrate: a body moves

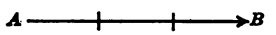


FIG. 61.

from west to east with a speed of 3 miles per hour. Draw the line AB, Fig. 61, 3 cm. long and from left to right. The line drawn 3 cm. long represents the magnitude, the speed, 3 miles per hour, and drawn from A to B represents the motion from left to right, or, from west to east. An arrow is placed upon the line indicating its sense, *i.e.*, from left to right.

A directed line, as AB, Fig. 61, is called a **vector.**

A vector quantity is also termed, for brevity, simply a **vector.** In what follows it is assumed that all problems are of such a nature that all vector quantities act in the same plane. The vectors representing these vector quantities may then be drawn upon a plane, the plane of the paper.

82. Vector Addition. In elementary algebra real numbers are usually represented graphically by points along a straight line. Thus, if O, Fig. 62, be any point upon the straight line $X'X$,

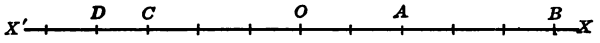


FIG. 62.

points to the right of O represent the positive numbers, while points to the left represent the negative numbers. To every real number there corresponds some point upon the line, and for every point of the line there exists a real number. If the points upon the line are chosen such that distances between them are proportional to the steps between the numbers which they represent, we have what was previously called a uniform scale. In what follows this uniform scale is assumed as the graphic representation of the real number system. The points A, B, C, D, etc., or the vectors OA, OB, OC, OD, etc., represent, respectively, the numbers 2, 5, −3, −4, etc.

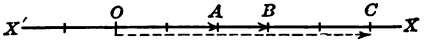


FIG. 63.

The addition of two real numbers, say 2 and 3, may be looked upon as consisting of the following operations: Move the vector OA, Fig. 63, parallel to itself; place it so that its initial point, which was originally at O, coincides with the terminal point, B, of the vector OB. Its new position is indicated by the dotted line BC. Draw the vector OC. It is easily seen, since the scale $X'X$ is uniform, that the vector OC represents the number 5, the sum of 3 and 2. It is readily seen that this process of addition holds for any set of real numbers (negative as well as positive), and that, in the process, either of the two vectors may be moved.

In general, let OB and OA, Fig. 64, be any two vectors. Move OA parallel to itself so that its initial point coincides with the terminal point, B, of the vector OB. It then takes up the position BC. Draw the vector OC. The vector OC is *defined* as the **vector sum** of the vectors OB and OA.

The **negative** of a vector is a vector of the same length, but extending in the opposite direction (having opposite sense).

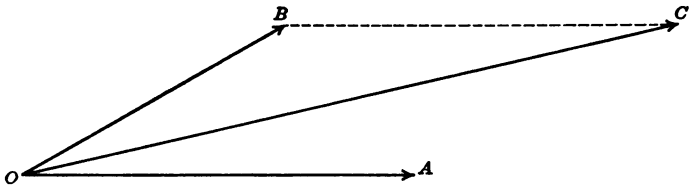


FIG. 64.

To subtract the vector OA from the vector OB means to find a third vector OC, Fig. 65, such that the sum of the vectors OC and OA is the vector OB. If OA is moved parallel to itself, so that its terminal point, A, falls upon the point B, and its initial point at C, the vector OC is such that the vector OC plus the vector OA equals the vector OB. Therefore,

vector OC = vector OB − vector OA

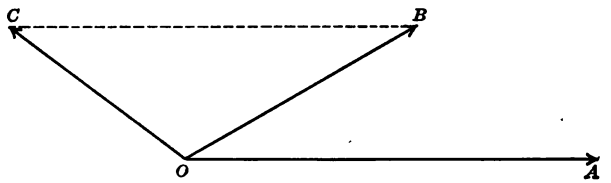


FIG. 65.

From the figure it is readily seen that, to subtract the vector OA from the vector OB is to add the negative of the vector OA, that is, AO, to the vector OB.

To find graphically the vector sum of any three vectors, find the sum of any two and to this sum add the third.

A vector is designated by giving its length, and the angle which it makes with the positive direction of the axis of x. Thus (10, 30°) when referring to a vector means that the vector is ten units long and its direction makes an angle of 30° with the positive direction of the axis of x.

Exercises

Find graphically the sum of the following vectors:

1. (2, 10°) and (3, 20°).
2. (5, 30°) and (10, 135°).
3. (6, 70°) and (5, 217°).
4. (5, 110°) and (3, − 71°).
5. (3, 30°), (7, 121°) and (11, − 75°).

83. Resultant Velocity. Let a raft move along a straight path with uniform speed of 2 feet per second; suppose a man

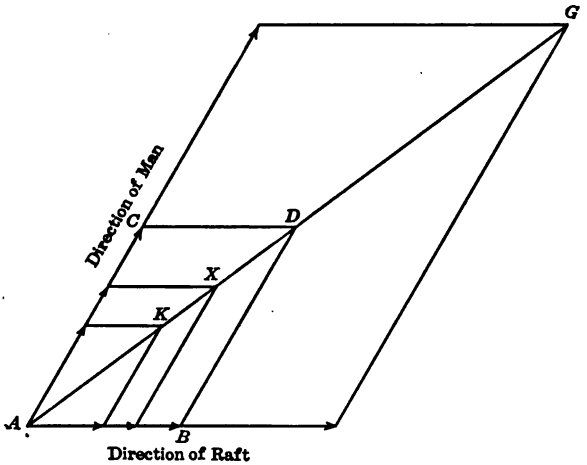


FIG. 66.

upon the raft walks at the rate of 3 feet per second in the direction indicated in Fig. 66. Let A be the point of the raft occupied by the man at some particular instant. In one second of time the point A will be translated by the movement of the raft to the point B. The length of the line AB represents 2 feet, and the direction of AB is the direction of motion of the raft. If

the raft were not moving, the man would walk along the line AC, and at the end of one second would be at the point C. The motion of the raft, however, has carried the point C to the point D; at the end of one second of time the man is at the point D instead of at the point C. $ABDC$ is a parallelogram. The sides AB and AC are vectors representing, respectively, the velocity of the raft relative to the earth and the velocity of the man relative to the raft. By the same reasoning it is seen that at the end of the second second the man will be at the point G. The points A, D, and G are upon a straight line. At the end of the first half second the man was at the point K, the center of the parallelogram $ABDC$, and at the end of any other fractional part of one second he was at some point X on the line AD. From similar triangles, it is easily shown that X is not only on the line AD, but at a position such that AX is to AD as the time interval is to one second. Thus, the man moves, relative to the earth, in the direction AD, with a uniform speed, and over a distance AD in one second of time. This is only another way of saying that the velocity of the man may be represented by the vector sum of the vector representing the velocity of the raft relative to the earth, and the vector representing the velocity of the man relative to the raft. In general, then, if a body is subjected to several velocities, its resultant velocity may be found by taking the vector sum of the vectors representing the component velocities.

84. Resultant of Concurrent Coplanar Forces Found Graphically. Experimentally, it is shown that the resultant of two concurrent[1] forces is such that it may be found in magnitude and direction by considering it the vector sum of the two vectors representing the two given forces. If a force of 10 pounds acts horizontally to the right, and if a second force of 6 pounds acts in a direction making an angle of 60° with the horizontal, the resultant of the two may be found graphically in the following manner. Lay off a line AB 5 inches long, Fig. 67. The vector AB repre-

[1] Forces acting upon a body are *concurrent* if they have the same point of application; otherwise they are *non-concurrent*.

Forces acting upon a body are *coplanar* if their lines of action lie in a plane; otherwise they are *non-coplanar*.

A single force, if it exists, which is equivalent to a system of forces is called the resultant of the system.

sents the 10-pound force. One inch in length represents a magnitude of two pounds. Draw the line AC 3 inches long making the angle BAC 60°. The vector AC represents the 6-pound force. Complete the parallelogram $ABDC$, or the triangle ABD, or the

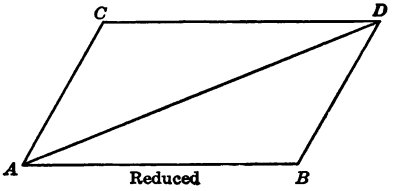


FIG. 67.

triangle ACD. AD is a vector representing the resultant of the 10- and the 6-pound forces. Measuring the length of AD it is found to be 7 inches, which shows that the magnitude of the resultant force is 14 pounds. Measuring the angle BAD it is found to be approximately 21° 45′, the direction angle of the resultant.

$ABDC$ is called the **parallelogram of forces.**

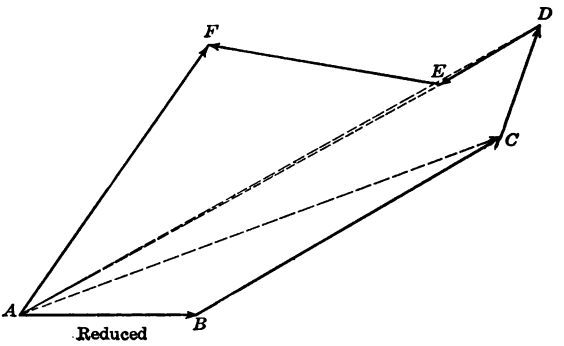


FIG. 68.

The resultant of three or more concurrent coplanar forces may be found by repeated application of the parallelogram of forces. Thus, to find the resultant of the five forces, P_1, P_2, P_3, P_4, P_5, whose magnitudes are, respectively, 3, 6, 2, 2, and 4 pounds, and

whose directions make, respectively, the angles 0°, 30°, 70°, 210° and 170° with the positive direction of the axis of x, proceed as follows: Draw the vector AB, Fig. 68, 3 inches long in a horizontal direction. Draw BC 6 inches long making an angle of 30° with the horizontal. AC is a vector representing the resultant of the forces P_1 and P_2. Draw a vector CD representing the force P_3. AD is a vector representing the resultant of P_1, P_2 and P_3. Draw the vector DE representing the force P_4. AE is the vector representing the resultant of P_1, P_2, P_3, and P_4. Draw EF, a vector representing the force P_5. AF is the vector representing the resultant of P_1, P_2, P_3, P_4, and P_5. AF is 5.59 inches in length, which shows that the magnitude of the resultant force is 5.59 pounds. The angle BAF measured with a protractor is found to be approximately 55°.

It is seen that in the construction, the lines AC, AD, and AE are entirely unnecessary and should be omitted. The figure $ABCDEFA$ is called the **polygon of forces.** AF, the resultant, is called the closing side. It will be noticed that the arrows on the vectors representing the given forces all run in the same sense around the polygon, while the arrow of the closing vector runs in the opposite sense.

If the vectors representing the given concurrent forces of themselves form a closed polygon, the resultant is zero, or the system of given forces is in **equilibrium.**

Exercises

Find graphically the resultant of the following concurrent, coplanar forces:

1. (3, 30°), (5, 45°), and (7, 190°).
2. (2, 0°), (3, 40°), (5, 170°), and (7, 135°).
3. (2, − 10°), (6, 90°), (5, 0°), and (7, 135°).
4. (3, 180°), (2, 270°), (5, 45°), and (1, 150°).
5. (1, 270°), (5, − 30°), (7, 90°), and (2, 30°).

85. Resultant of Concurrent Coplanar Forces found Analytically. Let AD, Fig. 69, represent a force both in magnitude and direction. If AB, be a line of any length, drawn in any direction, and if AB_1DC_1 be a parallelogram so constructed that AB_1 is a side and AD a diagonal, AD may be considered the resultant of two

forces represented by AB_1 and AC_1. Then in a process of analysis the force represented by AD may be replaced by the two forces represented by AB_1 and AC_1. The force AD is said to be resolved into the components AB_1 and AC_1. It is apparent that AD can be resolved into an infinite number of pairs of components; and if it were desirable, either, or both, of these components could be resolved further into two or more components.

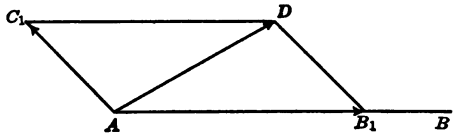


FIG. 69.

If the resolution is such that the components are parallel one to each axis of coördinates, the force is said to be resolved into its x- and y-components, or into its horizontal and vertical components.

It is seldom desirable to resolve a force into other than its x- and y-components, and this form of resolution is very helpful at times in finding the resultant force acting upon a body.

If OD, Fig. 70, represents a force acting in a direction making an angle θ with the positive direction of the axis of x, the magnitudes of the horizontal and vertical components are, respectively, $R \cos \theta$ and $R \sin \theta$, where R is the magnitude of OD. The vector representing the force in Fig. 70 is in the first quadrant; but the student will easily see, by sketching other figures, that the algebraic sign of the sin θ and cos θ will give to the component force the proper algebraic sign, *i.e.*, positive when up or to the right, and negative when down or to the left.

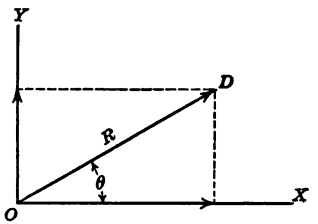


FIG. 70.

What is here called the *analytic method* of finding the resultant of a set of given forces, consists of the following process: resolve each force into its horizontal and vertical components, add algebraically the horizontal components, add algebraically the vertical components, and from these two sums find the resultant as indicated below.

It is required to find analytically the resultant of the five forces given in § 84. The work is put in tabular form:

	Magnitude	Angle	Cosine	Sine	X-component	Y-component
P_1	3	0°	1.000	0.000	3.000	0.000
P_2	6	30°	0.866	0.500	5.196	3.000
P_3	2	70°	0.342	0.940	0.684	1.880
P_4	2	210°	− 0.866	− 0.500	− 1.732	− 1.000
P_5	4	170°	− 0.985	0.174	− 3.940	0.696
		Sum	.....	.....	3.208	4.576

In the first column are placed the letters indicating the forces. In columns two and three are placed, respectively, the magnitudes and angles of these forces. In the fourth and fifth columns are placed, respectively, the cosines and sines of these angles. In the sixth column are placed the products of the magnitudes of the forces by the cosines of the angles. In the seventh column are placed the products of the magnitudes of the forces by the sines of the angles. In columns six and seven, then, are given, respectively, the x- and y-components of the five forces. Below are given the algebraic sum of the x-components and the algebraic sum of the y-components. The five forces may be considered as being replaced by two; one of 3.208 pounds acting to the right along the X-axis, the other of 4.576 pounds acting up along the Y-axis. It now remains to unite these two forces into one resultant, the resultant of the five forces. Since the two components of 3.208 and 4.576 pounds are mutually at right angles, the parallelogram of forces (See Fig. 70) is a rectangle; hence, the magnitude of the resultant $R = \sqrt{(3.208)^2 + (4.576)^2} = 5.59$. The direction of OC is given by $\tan\theta = \dfrac{4.576}{3.208} = 1.426$, or, $\theta = 54° 58'$.

Of the two values of the angle corresponding to $\tan\theta$, the student must be careful to select the correct value for θ, by noting the quadrant within which the resultant lies.

Exercises

1. Solve, analytically, placing work in tabular form, the exercises given in § 84.

86. The Simple Crane. To illustrate the application of the polygon of forces to mechanical problems, let us consider a simple crane. AB, Fig. 71, is the post, CE is the jib, and DE the rope attached to the jib at the point E. At the end of the jib is suspended a weight of 10 tons. It is required to find the tension in the rope and the compression in the jib. The point E is acted upon by three forces: one, of magnitude 10, acts down; one, of unknown magnitude, acts in the direction CE, and the third, of unknown magnitude, acts in the direction ED. Suppose CD = 12 feet,

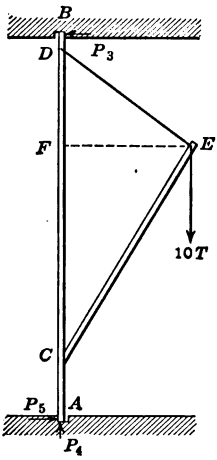


Fig. 71.

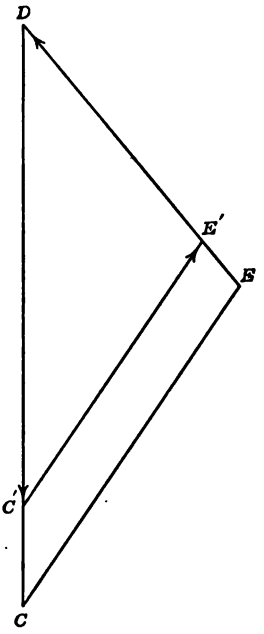


Fig. 72.

CE = 8 feet, and DE = 7 feet. Draw a triangle to scale with sides proportional to 12, 8, and 7, Fig. 72. Draw the side proportional to 12 vertical. Draw the side proportional to 7 through the upper end, and the side proportional to 8 through the lower end, of the vertical side. The triangle CED is drawn to scale. A vector representing the 10-ton force will point in the direction DC; a vector representing the compression (call it P_1) in the jib will

point in the direction CE, and a vector representing the tension (call it P_2) in the rope will point in the direction ED.

Draw the polygon of forces, $DC'E'$, for the three forces, 10, P_1, and P_2. This polygon must close if the point E is in equilibrium. Scale off the length $C'E'$, and the length $E'D$, which give, respectively, the magnitude of the compression in the jib and of the tension in the rope. If DC' is made 10 inches long, or 1 inch representing 1 ton, $C'E'$ is found to be approximately 6.67 inches and DE' approximately 5.83 inches. This gives a compression in the jib of 6.67 tons, and a tension in the rope of 5.83 tons.

Exercises

Find graphically the compression in the jib, and the tension in the rope, Fig. 71, if CD = 12 feet, CE = 8 feet, and ED = (*a*) 8 feet, (*b*) 9 feet, (*c*) 10 feet, (*d*) 11 feet, (*e*) 12 feet, (*f*) 13 feet, and (*g*) 14 feet. Plot two curves upon a sheet of rectangular coördinate paper, one representing the compression in the jib, the other representing the tension in the rope. Plot forces as ordinates, and lengths of DE as abscissas.[1]

87. Parallel Forces; Moment of Force. A force applied to a body may act throughout the whole, or a part, of its mass; or it may act over the whole, or a part, of its surface. A book lies upon the table. Gravitational force acts throughout its entire mass, whereas the force exerted upon it by the table acts only over that portion of its surface in contact with the table. If the surface over which the force acts is small compared with other dimensions, the force is treated as though it acted at a single point of the body. Thus, in Fig. 71, the post of the crane presses to the right,

[1] This problem may be solved more easily by making use of the theorem in geometry which says that homologous sides of similar triangles are in proportion. In Fig. 72, the triangles DCE and $DC'E'$ are similar, and

$$\frac{DC'}{DC} = \frac{C'E'}{CE} = \frac{DE'}{DE},$$

or

$$\frac{10}{12} = \frac{C'E'}{8} = \frac{DE'}{7},$$

or

$$C'E' = 6\tfrac{2}{3}$$
$$DE' = 5.833$$

Solve the above exercises by this method.

against the ceiling, and the ceiling presses against the crane with a force of equal magnitude, but in the opposite direction. This force, P_3, is distributed, uniformly or non-uniformly, over the entire surface of the post in contact with the ceiling; but this bearing surface is so small compared with other dimensions of the crane that the force P_3 is considered acting, and is said to act, at a point.

A weight of 10 tons is suspended from the end of the jib. Its line of action is vertical and passes through the point E. If the cable suspending the weight were two, three, or any number of times as long, the weight would still exert the same influence upon the crane. For the cable still pulls upon the crane with a force of 10 tons and in a downward direction. The line of action remains the same, *i.e.*, vertical and through the point E. In considering forces acting upon a body, the position of the line of action, and not the exact point of application of the force, is what enters into the solution of the problem.

It is readily seen that *the line of action of the resultant of two forces passes through the point of intersection of their lines of action.*

In finding the resultant of two parallel forces it becomes necessary to consider two cases: when the forces act in the same direction and when the forces act in opposite directions.

88. Two Parallel Forces Acting in the Same Direction. Let AP, Fig. 73, be a vector representing a force, P, whose line of action passes through the point A. Let BQ be a vector representing a force, Q, whose line of action passes through the point B. P and Q are parallel forces acting upon a body in the same direction. The resultant of P and Q together with its line of action is to be found. At the points A and B insert two forces, S and S', having the same line of action, equal in magnitude but acting in opposite directions. The resultant of these two forces is zero, hence their introduction in no way changes the resultant force acting upon the body. If AS and BS' represent the forces S and S', AP' represents P', the resultant of P and S; and BQ' represents Q', the resultant of Q and S'. Since S and S' balance, the resultant of P' and Q' is the resultant of P and Q. The lines of action of P' and Q' intersect at C. At the point C resolve the force P' (represented by CP'') into its original components P

and S, represented by CP''' and CS'', respectively. At the point C resolve Q' (represented by CQ'') into its original components Q and S', represented by CQ''' and CS''', respectively. The forces S and S' again balance, leaving the resultant of P and Q equal to the vector sum of CQ''' and CP'''. From similar triangles it is seen that CQ''' is equal and parallel to BQ, and that CP''' is

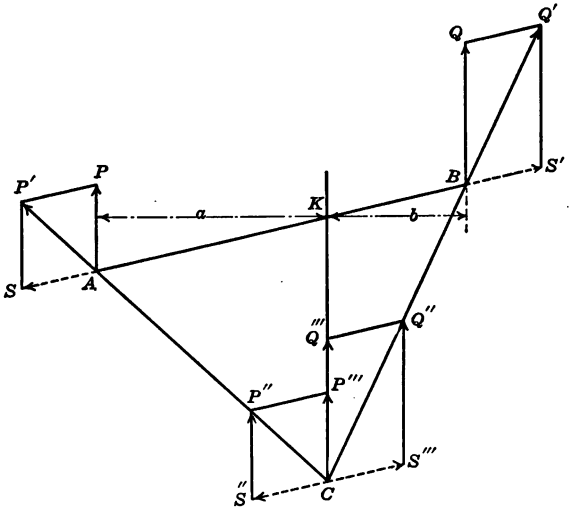


FIG. 73.

equal and parallel to AP. Hence, the resultant of P and Q is $P + Q$, whose line of action passes through the point C and is parallel to the lines of action of P and Q.

From similar triangles:

$$\frac{AP}{AS} = \frac{CK}{AK} \text{ and } \frac{BQ}{BS'} = \frac{CK}{BK}.$$

From these two equations, and since $AS = BS'$, it follows that

$$\frac{AP}{BQ} = \frac{BK}{AK}.$$

If a is the distance of K from the line of action of P, and if b is

the distance of K from the line of action of Q, it follows from similar triangles that

$$\frac{b}{a} = \frac{BK}{AK}.$$

Uniting this equation with the next preceding,

$$\frac{AP}{BQ} = \frac{b}{a},$$

or

$$\frac{P}{Q} = \frac{b}{a},$$

or

$$Pa = Qb.$$

The product of the magnitude of a force by the distance of its line of action from a fixed point is called the **moment** of the force about the point. The point about which moments are taken is called the **origin of moments,** and the perpendicular distance from the origin to the line of action of the force is called the **arm of the force.**

If the relation of the direction of the force to the origin of moments is such as to suggest positive rotation about the origin, the

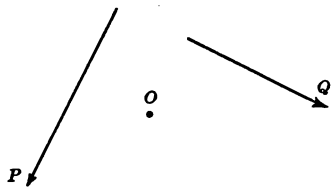


FIG. 74.

moment is considered **positive.** If the relation is such as to suggest negative rotation, the moment is considered **negative.** Thus, the moment of the force P about the point O, Fig. 74, is positive; the moment of the force Q about O is negative.

The moment of the force Q about the point K, Fig. 73, is $Q\,b$; the moment of the force P about the same point is $-P\,a$. The (algebraic) sum of the moments is $Q\,b - P\,a$, but since in magnitude $P\,a = Q\,b$, this sum is zero. In other words, the sum of the

moments of two parallel forces acting in the same direction about any point in the line of action of their resultant is zero.

Let O, Fig. 75, be any point in the plane. Let c be its perpendicular distance from the line of action of R, the resultant of the parallel forces P and Q; and d its perpendicular distance from the

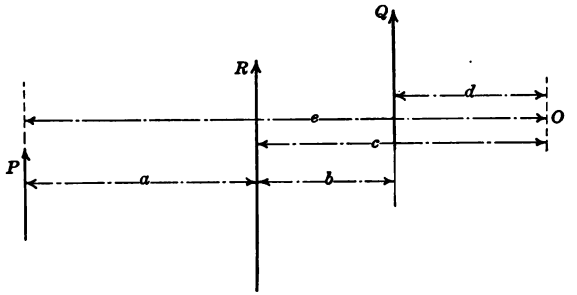


FIG. 75.

line of action of the force Q; and e its perpendicular distance from the line of action of the force P. Then

$$c = e - a$$

$$Rc = Re - Ra$$

and since $R = P + Q$,

$$Rc = Pe + Qe - Pa - Qa$$

and since $Pa = Qb$,

$$Rc = Pe + Qe - Qb - Qa$$

$$Rc = Pe + Q(e - b - a)$$

and since $e - b - a = d$,

$$Rc = Pe + Qd.$$

This equation shows that the moment, about any point, of the resultant of two parallel forces acting in the same direction is equal to the sum of the moments of the two forces about the same point. The proof given above considers the lines of action of the two given forces, and the line of action of their resultant, to be upon one side of the point O. The student may easily show that the same law holds when the point O is taken between the lines of action of the two given forces, if by sum is meant algebraic sum.

89. Two Parallel Forces Acting in Opposite Directions. Let P and Q, Fig. 76, be two parallel forces acting in opposite directions. By analysis similar to that of the preceding section, the student will show: that the resultant of the two forces is the algebraic sum of P and Q; that its line of action is parallel to that of P and Q, and so placed that the moment of the resultant about any point is the algebraic sum of the moments of P and Q about the same point.

The above proof fails when P and Q are equal in magnitude, for then AP' and BQ' are parallel and the point C is at infinity. In

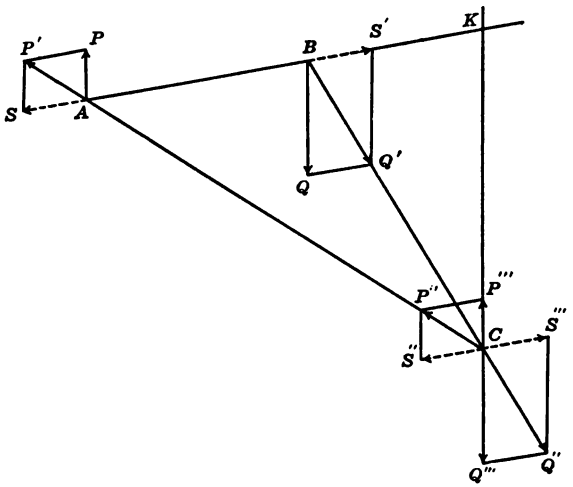


FIG. 76.

this case the algebraic sum of the two forces is zero, and they cannot be replaced by a single force whose effect upon the body is equivalent to that of the two forces.

A system of two non-collinear parallel forces, equal in magnitude, and acting in opposite directions, is called a **couple.** The product of the magnitude of the forces by the perpendicular distance between their lines of action is called the **moment of the couple.** If the couple tends to produce positive rotation its moment is positive; if negative rotation, its moment is negative.

The student will show that the moment of a couple is equal to the algebraic sum of the moments of the two forces of the couple about any point. He will also show that the algebraic sum of the moments of two equal parallel forces acting in opposite directions is the same for all origins of the moments.

90. A System of Parallel Forces is Reducible Either to a Single Force or to a Single Couple. In a system of three or more parallel forces there must be at least two that do not form a couple. These two forces may be replaced by their resultant force, as was shown in §§ **88** and **89**. The magnitude of the resultant is the algebraic sum of the two components and its moment about any point is the algebraic sum of the moments of the two forces about the same point. In this way the system of forces is replaced by another system having a number of forces less by one. The algebraic sum of the forces and the algebraic sum of thier moments about any point remain, however, the same.

By continuing this process of reducing the number of forces, the system is eventually replaced by one consisting of only two parallel forces. This system, if not a couple, can be replaced by a single force, the resultant of the two forces. *Thus, the resultant of a system of parallel forces is either a couple or a single force.*

The resultant can be a single couple only when the algebraic sum of the given forces is zero, and when the algebraic sum of the moments of all the given forces about any point is different from zero. For, by the above method of reduction of the number of forces of the system one at a time, the algebraic sum of the magnitudes of the forces remains unchanged, and since this sum is zero when a couple is obtained it must have been zero at the beginning; the algebraic sum of the moments about any point remains unchanged, and since the moment of the couple cannot be zero, the sum of the moments of the given forces cannot be zero.

91. System of Parallel Forces in Equilibrium. A system of forces applied to a body are said to be **in equilibrium** if the state of motion of the body is unaltered by the application of the forces.

If the algebraic sum of a system of parallel forces is not zero, the resultant of the system is a single force and the system is not in equilibrium. One of the *necessary* conditions for equilibrium of a system of parallel forces is, then, that the algebraic sum of the

forces be zero. If the algebraic sum of the moments of the forces about any point is not zero while the algebraic sum of the forces is zero, the resultant is a couple and the body is not in equilibrium.

The necessary and sufficient conditions for a system of parallel forces to be in equilibrium are: The algebraic sum of the forces must be zero; and the algebraic sum of the moments of the forces about any point must be zero.

The above law, it will be noticed, includes, as a special case, the law of the lever.

Exercises

1. A lever 14 feet long is used for raising a weight of 500 pounds. Where may the fulcrum be placed if a downward pressure of 100 pounds is available at the other end of the lever?

2. The two end clevis holes of a double-tree are 14 inches and 20 inches from the middle clevis hole. The team is pulling 300 pounds. How much does each horse pull?

3. What must be the ratio of the longer arm to the shorter arm of a double-tree: (*a*) if one horse is to pull 5 percent more than the other horse; (*b*) if one horse is to pull 5 percent less than the other horse? If the team pulls 300 pounds, how much does each horse pull in each of the above cases?

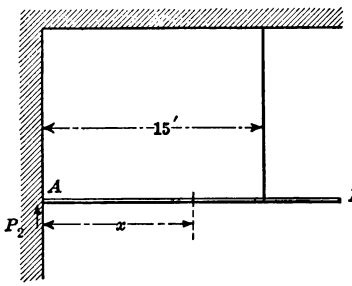


FIG. 77.

4. A horizontal bar, AB, 30 feet long weighs 30 pounds per linear foot. At the point A is suspended a weight of 10 tons, and at the point B is suspended a weight of $3\frac{1}{2}$ tons. Find the magnitude, the direction, and the line of action of the resultant of the two weights and the gravitational forces acting upon the bar.

HINT: The line of action of the gravitational forces acting upon the particles of the bar passes through the geometric center of the bar.

5. Let AB, Fig. 77, be a plank hinged at the point A, and held in a horizontal position by a vertical rope. The plank is 20 feet long and weighs 110 pounds. The rope is fastened to the plank 15 feet from the hinge. Find the tension in the rope when a man weighing 150 pounds stands upon the plank x feet from the hinge.

HINT: If P_1 represents the tension in the rope, taking moments about A as origin gives $15P_1 - (110)(10) - 150x = 0$, or $P_1 = 10x + 220/3$. For, the system of parallel forces is in equilibrium, and the algebraic sum of the moments about any point must be zero. The above equation, which is linear, expresses the relation between the tension in the rope and the distance of the man from the point A. When the man is at the point A, the tension is $73\frac{1}{3}$ pounds. When he is at the point B, the tension is $273\frac{1}{3}$ pounds. $P_2 + P_1 - 110 - 150 = 0$, or $P_2 = 360 - P_1$, or $P_2 = 286\frac{2}{3} - 10x$, which gives the reaction at the hinge. For, by § **90** the sum of the parallel forces must be zero. When the man is at the point B, $P_2 = 86\frac{1}{3}$ pounds.

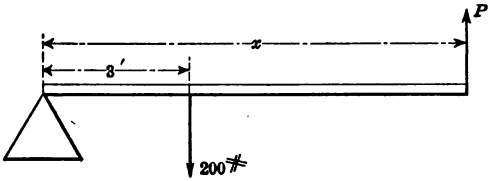


FIG. 78.

6. A weight of 200 pounds is to be raised by a lever x feet long, Fig. 78. The distance from the weight to the fulcrum is 3 feet and the lever weighs 4 pounds to the foot. If the lever is very short there is but little advantage gained by its use, whereas if it is very long its increased weight will more than compensate the gain in leverage. Evidently there is some length which will make the force P a minimum. Show that $P = \dfrac{2(300 + x^2)}{x}$. Find the values of P when x equals 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, $17\frac{1}{2}$, 18, 19, and 20. Plot results upon a sheet of rectangular coördinate paper.

HINT: When substituting in the formula, place it in the form

$$P = 2\left(\frac{300}{x} + x\right)$$

7. The end holes A and B of a double-tree, Fig. 79, are $2a$ inches apart. The center hole C is placed d inches in front of the mid-point of AB. If one horse is pulling ahead of the other as indicated in Fig. 79, find what percent more the rear horse is pulling than the forward horse.

HINT:

$$P_1(a \cos \alpha - d \sin \alpha) = P_2(a \cos \alpha + d \sin \alpha)$$

$$\frac{100(P_1 - P_2)}{P_2} = \frac{200d \sin \alpha}{a \cos \alpha - d \sin \alpha}$$

8. Same as exercise 7, but with $a = 13$ inches, $d = 3$ inches, and $\alpha = 30°$.

9. With the statement of exercise 7, the difference between P_1 and P_2 is what percent of the load?

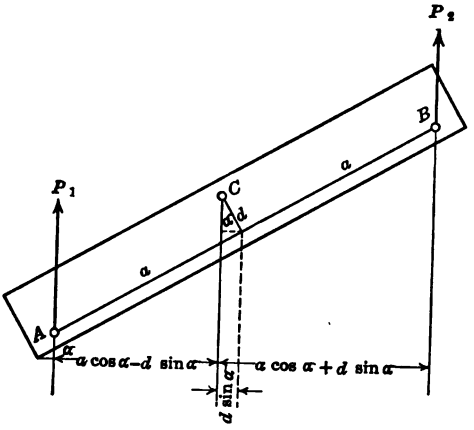


FIG. 79.

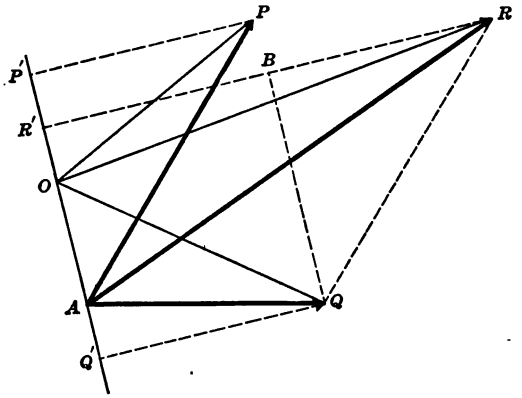


FIG. 80.

92. The Sum of the Moments of Two Non-parallel Forces. Let AP and AQ, Fig. 80, represent two forces. Let AR represent their resultant R. The moment of P about O is the magnitude of

P, (AP), multiplied by the perpendicular distance from O to the line of action of P. This moment is then represented by twice the area of the triangle OAP. Similarly, the moment of Q about O is represented by twice the area of the triangle OAQ; and the moment of R by twice the area of the triangle OAR. Draw PP', RR', and QQ' perpendicular to AO; and QB perpendicular to RR'.

The area of the triangle $OAP = \frac{1}{2}(PP' \cdot AO)$.

The area of the triangle $OAQ = \frac{1}{2}(QQ' \cdot AO)$.

The area of the triangle $OAR = \frac{1}{2}(RR' \cdot AO)$.

$$RR' = RB + BR',$$

or

$$RR' = PP' + QQ'.$$

By uniting these equations

$$2\,(\text{area of } \Delta\, OAP) + 2\,(\text{area of } \Delta\, OAQ) - 2\,(\text{area of } \Delta\, OAR) = 2(PP' + QQ' - RR') = 0,$$

which shows that *the moment of the resultant, R, is equal to the sum of the moments of P and Q.*

In the above proof, the point O was so located that the algebraic signs of the three moments were the same. The student will prove the formula for the case in which O is taken between the sides of the angle QAP. When this has been done it will be seen that "sum" in the above law means *algebraic sum.*

From the above theorem and from what has been proved regarding moments of parallel forces, it will be seen that *the algebraic sum of the moments of any number of forces is equal to the moment of their resultant, and the algebraic sum of the moments of any number of forces in equilibrium is zero.*

A restatement of the above conclusions follows: The resultant of a system of coplanar forces (not in equilibrium) is either a single force or a couple. If the resultant is a single force, its magnitude and direction may be found graphically by finding the sum of the vectors representing the given forces; its line of action is such that its moment about any point is the algebraic sum of the moments of the several forces about the same point. If the resultant is a couple, the vector sum of the vectors representing the forces is zero, and the algebraic sum of the moments of the several forces about any point is equal to the moment of the couple. For

a system of coplanar forces to be in equilibrium, *the vector sum of the vectors representing the several forces must be zero, and the algebraic sum of the moments of the several forces about any point must be zero.* The first condition throws out the possibility of the resultant being a single force. The second condition throws out the possibility of the resultant being a couple.

93. Equations of Equilibrium. In the preceding sections it was shown that if a system of coplanar forces is in equilibrium, the algebraic sum of the moments of the forces about any point must be zero. This law is generally written in the form of an equation:

$$\Sigma Pa = 0$$

P represents the magnitude of force, a its moment arm, and Σ (the Greek letter Sigma) is a symbol meaning to take the algebraic sum. Another condition for equilibrium is that the vector sum of the vectors representing the given forces must be zero. Another way of stating this law is that if the given forces were considered concurrent, their resultant would be zero. It was shown, however (§ **85**), that one method of finding the resultant of a system of concurrent coplanar forces is first to resolve each force into its x- and y-components, and from the algebraic sum (X) of the x-components and the algebraic sum (Y) of the y-components compute the magnitude of the resultant (R) by the formula $R = \sqrt{X^2 + Y^2}$. If R is zero, X and Y separately equal zero. This law is generally written in the form of two equations:

$$\Sigma X = 0$$

and

$$\Sigma Y = 0$$

The symbol Σ(the Greek letter Sigma) means to take the algebraic sum. X in the above equation means x-component. ΣX then means the algebraic sum of the x-components. ΣY means the algebraic sum of the y-components.

The equations

$$\Sigma X = 0$$
$$\Sigma Y = 0$$

and

$$\Sigma Pa = 0$$

are called the equations of equilibrium for a system of coplanar forces.

The application of the equilibrium equations will be illustrated by examples:

EXAMPLE 1: The crane, Fig. 71, considered as a whole, is acted upon by four forces: viz., the downward force of 10 tons, the upward push of the floor, P_4; the push of the floor to the right, P_5; and the push of the ceiling to the left, P_3. P_4, P_5, and P_3 are unknown in magnitude, and are to be found. Let $AB = 12$ feet and $FE = 4$ feet. Then

$$\Sigma X = P_5 - P_3 = 0$$
$$\Sigma Y = P_4 - 10 = 0$$

and by taking moments about the point A,

$$\Sigma Pa = 12P_3 - 10 \cdot 4 = 0,$$

or $P_3 = 10/3$. The forces P_4 and P_5 contribute nothing to the moment equation, for they act through the point A about which moments are taken; hence their moment arms and moments are zero.

The second equation gives $P_4 = 10$ tons.

The first equation gives $P_3 = P_5 = 10/3$ tons. This means that the post of the crane pushes to the left against the flange of the floor with a force of $3\frac{1}{3}$ tons. The construction at this point, as well as at the ceiling, must be such as to withstand this pressure.

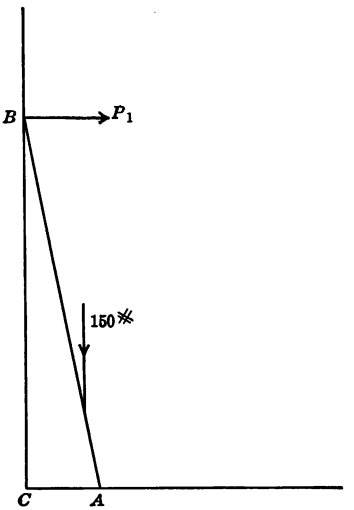


FIG. 81.

EXAMPLE 2: Let AB, Fig. 81, be a ladder 16 feet long, resting against a smooth vertical wall, CB. The lower end of the ladder rests upon the horizontal ground, 6 feet from the building. CB is 14.83 feet long. The ladder weighs 35 pounds, and a man

who ascends weighs 150 pounds. Find the pressure of the upper end of the ladder against the wall when the man stands on a rung 5 feet from the point A.

By taking moments about the point A,

$$\Sigma Pa = +35{\cdot}3 + 150{\cdot}15/8 - 14.83\,P_1 = 0,$$

or $P_1 = 26.04$ pounds. (The weight of the ladder is considered as a single force acting through the middle point.)

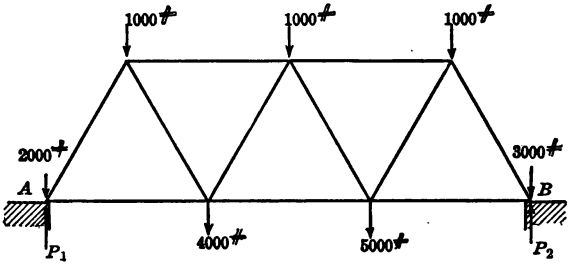


FIG. 82.

Taking moments about the point A, letting x be the distance (measured along AB) the man stands from the point A,

$$\Sigma Pa = 35{\cdot}3 + 150\,\frac{6x}{16} - 14.83\,P_1 = 0,$$

or

$$P_1 = \left[\frac{225}{4}\,x + 105\right] \div 14.83.$$

From this equation P_1 can be calculated for all values of x, *i.e.*, for any position of the man upon the ladder. The equation is linear and hence its graphic representation upon a sheet of squared paper is a straight line. In order to draw the straight line all that is needed are the coördinates of two points. These may be found by giving to x two values, as 0 and 16, and calculating the corresponding values of P_1.

EXAMPLE 3: Let all members of the bridge truss, Fig. 82, be equal in length, k feet. The bridge sustains loads as indicated. Find the reactions, P_1 and P_2, at the points A and B. By taking moments about the point A, it follows that

$$3k\,P_2 - 1000\frac{k}{2} - 4000\,k - 1000\frac{3k}{2} - 5000{\cdot}2k - 1000\frac{5k}{2} - 3000{\cdot}3k = 0,$$

or

$$3\,P_2 - 500 - 4000 - 1500 - 10{,}000 - 2500 - 9000 = 0,$$

or

$$3\,P_2 - 27{,}500 = 0,$$

or

$$P_2 = 9166\tfrac{2}{3} \text{ pounds.}$$

$$\Sigma Y = P_1 + P_2 - 2000 - 1000 - 4000 - 1000 - 5000 - 1000 - 3000 = 0,$$

or

$$P_1 + P_2 = 17{,}000,$$

or

$$P_1 + 9166\tfrac{2}{3} = 17{,}000,$$

or

$$P_1 = 7833\tfrac{1}{3}.$$

Exercises

1. Find the magnitudes of the forces P_1, P_2, and P_3, Fig. 83, neglecting the weight of the crane.

2. Find the magnitudes of the forces P_1 and P_2, Fig. 84, also the tension in the rope.

HINT: Draw ABC to scale, using 1/2 inch to represent a foot. Neglect the weight of AC.

[94.] Tension and Compression. The tension or compression in members of a construction may be easily computed, if the weights of the members are neglected, if the loads are applied at the joints, and if the members are discontinuous at the joints. By discontinuous is meant that a single piece does not form a side of more than one polygon. Thus, in the roof truss, Fig. 85, the tie AB consists of three pieces, AD, DE, and EB, bolted together at the points D and E.

If the loads are applied at the joints, the tension or compression in every member is the same throughout the entire length of that member, and the line of action of the compressive or tensile force within the member is that of the longitudinal dimension of that member.

The method of finding tension and compression will be illustrated by solving the problem represented by Fig. 85.

Let $AD = DE = EB = 10\frac{2}{3}$ feet

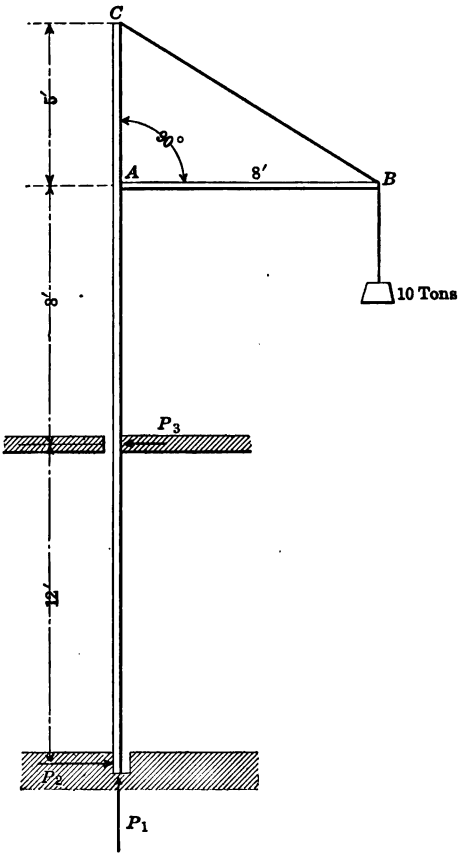


FIG. 83.

Let $CK = 8$ feet

$AC = BC = 17.88$ feet

and $CD = CE = 9.61$ feet

To find the reactions, P_1 and P_2, of the supports use $\Sigma Y = 0$, and $\Sigma Pa = 0$, taking moments about the point A.

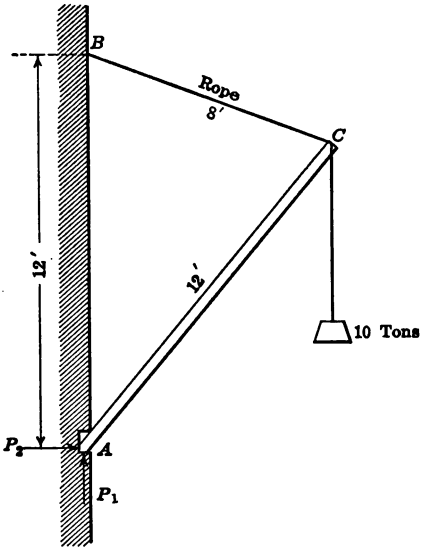


FIG. 84.

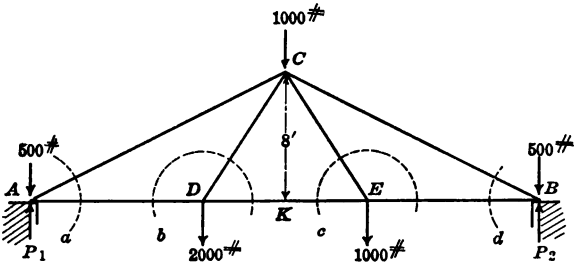


FIG. 85.

$$\Sigma Y = P_1 + P_2 - 500 - 2000 - 1000 - 1000 - 500 = 0,$$

or

$$P_1 + P_2 = 5000.$$

$$\Sigma Pa = 2000 \cdot 10\tfrac{2}{3} + 1000\,16 + 1000 \cdot 21\tfrac{1}{3} + 500 \cdot 32 - P_2 \cdot 32 = 0,$$

or

$$P_2 \cdot 32 = 74{,}667,$$

or

$$P_2 = 2333 \text{ pounds.}$$

Then

$$P_1 = 2667 \text{ pounds.}$$

Imagine the members AC and AD cut by the section a and the right-hand portion removed. In order to keep the left-hand portion in equilibrium, Fig. 86, a, forces P_3 and P_4, equal in magnitude to the tension or compression in AC and AD, must be introduced

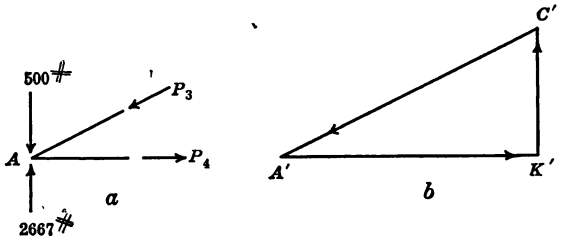


FIG. 86.

upon the cut ends. To express this in another way: suppose an imaginary plane through the rafter AC divide it into an upper and a lower part. The part below pushes against the part above, and the part above pushes against the part below; the two compressive forces are equal in magnitude and this magnitude is called the compression within the rafter. If the upper part of the member were removed, and if a force equal to the compression were introduced acting upon the remaining cut end of AC and in a direction toward A, the lower part of the rafter would be acted upon by the same force as before the upper portion was removed.

The four forces acting upon the remaining portion of the truss, Fig. 86, a, are in equilibrium. The two vertical forces acting at the point A may be considered as one force acting up, of magnitude 2167 pounds. Draw the line $C'K'$, Fig. 86, b, to scale, representing the upward 2167-pound force. $C'K'$ is parallel to CK, Fig. 85. Through C' draw a line parallel to AC and through K' draw a line parallel to AK. These two lines intersect at A'.

$K'C'A'$ is then the force polygon for the three forces, 2167, P_3 and P_4. $C'A'$ is the vector representing the force P_3, and $A'K'$ is the vector representing the force P_4. The magnitudes of P_3 and P_4 may be scaled off from this triangle. Since $K'C'A'$ is a closed polygon, the arrows run around the polygon in the same sense. The arrow of $C'A'$ then points from C' toward A', which says that the arrow of P_3, Fig. 86, a, was chosen to point in the proper direction, or the member AC is in compression. Similarly, the arrow of P_4 was chosen in the proper direction, or the member AD is in tension.

The magnitudes of P_3 and P_4 may be found by another method which does not require drawing the triangle $K'C'A'$ to scale. The triangles $K'C'A'$ and KCA, Fig. 85, are similar, and

$$\frac{C'A'}{CA} = \frac{K'C'}{KC},$$

or

$$\frac{C'A'}{17.88} = \frac{2167}{8},$$

or

$$C'A' = 4842,$$
$$P_3 = 4842 \text{ pounds.}$$

Again:

$$\frac{A'K'}{AK} = \frac{K'C'}{KC},$$

or

$$\frac{A'K'}{16} = \frac{2167}{8},$$

or

$$A'K' = 4333,$$
$$P_4 = 4333 \text{ pounds.}$$

To find the tension or compression in the members CD and DE, consider the portion about the point D removed by the cut b, Fig. 85.

At the cut ends insert the forces P_5 and P_6, Fig. 87, *a*, equivalent to the tension or compression in the cut members. The force polygon for the four forces is represented by the triangle $K'C'D'$, Fig. 87, *b*. This triangle is similar to triangle KCD, Fig. 85.

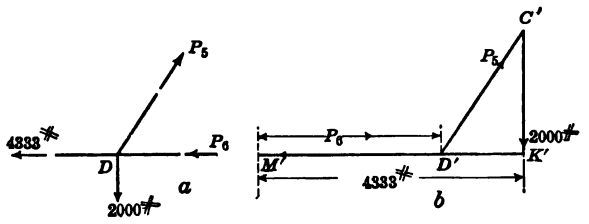


FIG. 87.

Hence

$$\frac{D'C'}{DC} = \frac{C'K'}{CK},$$

or

$$\frac{D'C'}{9.61} = \frac{2000}{8},$$

or

$$D'C' = 2402.5,$$

$$P_5 = 2402.5 \text{ pounds tension.}$$

Again:

$$\frac{K'D'}{KD} = \frac{2000}{8},$$

or

$$\frac{M'D' + 4333}{5\frac{1}{3}} = \frac{2000}{8},$$

or

$$M'D' = -3000 \text{ pounds.}$$

The negative sign shows that the arrow of P_6 points in the wrong direction.

$$P_6 = 3000 \text{ pounds tension}$$

In a similar way by means of cuts *c* and *d*, Fig. 85, the tension or compression in *EB*, *BC*, and *EC* may be found.

Exercises

1. Find the tension or compression in the members *BC*, *BE*, *EC*, and *ED*, of the truss given in Fig. 85.

2. Find the tension or compression in each member of the truss given in Fig. 82.

95. Rope and Pulley. Let P_1 and P_2, Fig. 88, be the tensions in a rope upon opposite sides of a pulley over which it passes. If

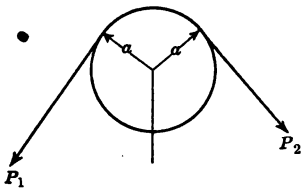


FIG. 88.

there be no friction between the pulley and the axle, the line of action of the force exerted by the axle upon the pulley will pass through the center of the pulley. Taking moments about the center of the wheel (calling a the radius of the wheel)

$$P_1 a - P_2 a = 0, \quad \text{or } P_1 = P_2$$

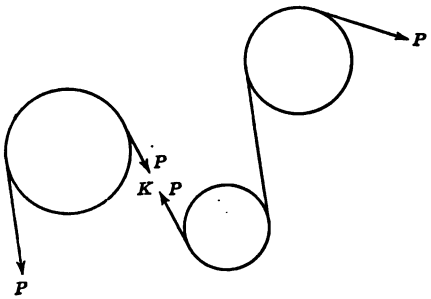


FIG. 89.

This says that if a flexible rope passes over a series of frictionless pulleys, the tension at every point of the rope is the same. In Fig. 89, the pull on each end of the rope is the same; and if the

rope be cut at any point, K, a force equal to P must be introduced upon each free end of the rope in order to hold the system in equilibrium.

Fig. 90 represents a system of pulleys by which a pull, P, holds a weight of 10 tons in equilibrium. To find the tension in the rope passing around pulley E, imagine the rope cut by the section

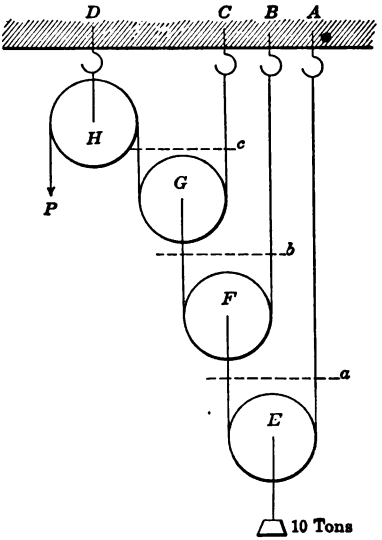


FIG. 90.

a, and forces introduced upon the cut ends equal to the magnitude of the tension in the rope. Represent these forces by P_1. The forces acting upon the part of the body below the section are in equilibrium, and $P_1 + P_1 - 10 = 0$, or $P_1 = 5$. The tension in the rope passing about pulley E is 5 tons.

If a second cut b be made, and if P_2 represent the tension in the rope passing about pulley F, $P_2 + P_2 - P_1 = 0$, or $P_2 + P_2 - 5 = 0$, or $P_2 = 2\frac{1}{2}$. The tension in the rope passing about pulley F is $2\frac{1}{2}$ tons.

If a third cut c be made, and if P represents the tension in the rope passing around pulleys G and H, $P + P - P_2 = 0$, or $P + P - 2\frac{1}{2} = 0$, or $P = 1\frac{1}{4}$. The pull, P, necessary to hold the weight 10 tons in equilibrium is $1\frac{1}{4}$ tons.

Exercises

1. Find the pull P which will hold, by means of the system of pulleys shown in Fig. 91, the weight of 10 tons in equilibrium. Find the pull on the hooks A, B, and C.

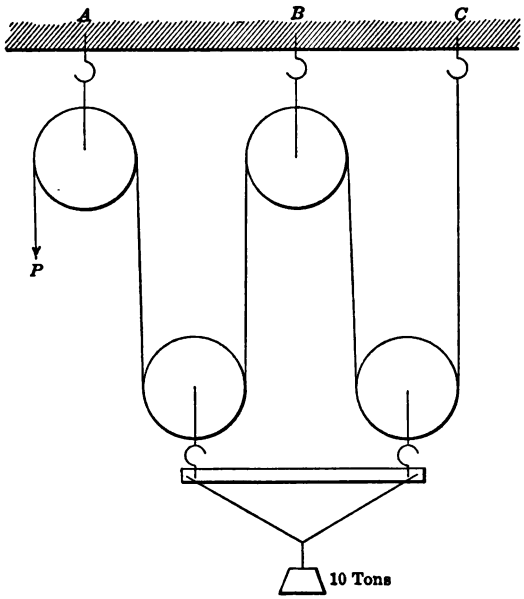


FIG. 91.

2. Find the pull P which will hold, by means of the system of pulleys shown in Fig. 92, the weight of 10 tons in equilibrium. Find the pull on the hook A.

3. The differential wheel and axle shown in Fig. 93 consists of three drums of radii a, b, and c. A force P is applied to a rope which unwinds from the largest drum. As the drums rotate a rope unwinds from the

smallest and winds up upon the medium sized drum, lifting a weight of 10 tons. Find P which will hold the weight in equilibrium. As P moves down x feet, how far does the weight rise?

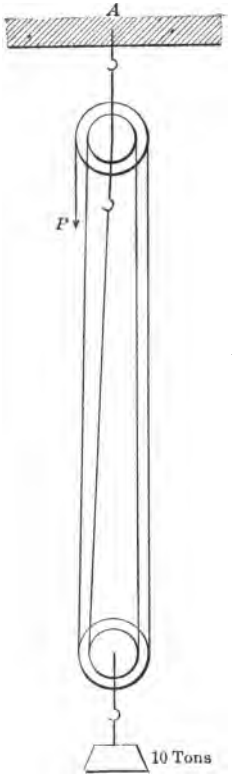


FIG. 92.

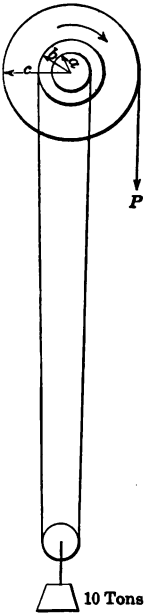


FIG. 93.

CHAPTER VII

PERMUTATIONS, COMBINATIONS, AND THE BINOMIAL EXPANSION

96. A Fundamental Principle. From a recitation room there are two doors leading into the hall; from the hall there are three exits. In how many ways can a person leave the building from the room? Let the doors of the room be numbered 1 and 2, and let the exits from the hall be marked A, B, and C. A person may pass through door 1 and exit A; door 1 and exit B; door 1 and exit C; door 2 and exit A; door 2 and exit B; or door 2 and exit C. After passing through door 1 there are three possible paths: one through exit A; one through exit B; and one through exit C. There are exactly the same number of paths passing through door 2. In all there are 2×3 or 6 ways of passing from the building.

If there were m doors leading from the room into the hall, and n exits from the hall, obviously there would be $m \times n$ ways of leaving the building.

The following is a fundamental principle: *If one thing can be done in m ways, and if a second can be done in n ways, the two things can be done in the order indicated in* $m \times n$ *ways.*

It is obvious that the order in which the two things are done does not affect the number of ways in which they may be done. They may even be done simultaneously. To illustrate: suppose a penny and a die are thrown simultaneously; the penny will fall in one of two ways; the die will fall in one of six ways; then, the two will fall simultaneously in one of 2×6, or 12 ways. This illustration shows that the *fundamental principle* applies not only to the number of ways of doing two things but also to the number of ways two events may occur.

It will be seen at once that if one thing can be done in m_1 ways, a second in m_2 ways, a third in m_3 ways, and so on, the number of ways they can be done together is $m_1 \times m_2 \times m_3 \times \ldots$

97. Permutation and Combination Defined. Suppose we have five cubes, one painted white, one black, one red, one blue, and one green. From these five a selection of one is made. This can be done in five ways. Each time a selection of one is made, a selection of four is left consisting of a different combination of colors.

A selection of two cubes is made. This may be done in ten ways, as:

white and red,	black and blue,
white and black,	black and green,
white and blue,	red and blue,
white and green,	red and green,
black and red,	blue and green.

Each time a selection of two cubes is made, a selection of three cubes remains. Then a selection of three cubes from the five cubes may be made in ten ways. A selection of four cubes may be made in five ways, and a selection of five cubes may be made in one way.

If from n things, a group of r ($r \leq n$) are selected, the selection is called **a combination of the n things taken r at a time.** The symbol representing the number of combinations that can be made of the n things taking r at a time is ${}_nC_r$. Thus, from the previous illustrations, ${}_5C_1 = 5$; ${}_5C_2 = 10$; ${}_5C_3 = 10$; ${}_5C_4 = 5$, and ${}_5C_5 = 1$.

Consider again the five colored cubes. Select two by first picking out one and then a second. This may be done in twenty ways if the order of the selection is considered (the selection first black, then white, is considered different from the selection first white, then black). For, by the fundamental principle, the first selection may be made in five ways, and the second selection in four ways; the total number of selections possible is $4 \times 5 = 20$.

Any selection of r ($r \leq n$) things taken from n things which takes into consideration the order of selection is called a **permutation of n things taken r at a time.**

The symbol representing the number of permutations of n things taken r at a time is ${}_nP_r$.

It is to be remembered that with permutations we consider the order of selection, or the arrangement, of the objects as well as the

individuals which go to make up the selection, while with combinations we consider only the individuals of the selection without reference to the order of their selection or arrangement. To illustrate: If in a plane there are five points, no three of which are upon one straight line, the problem of finding the number of lines determined by the points is a problem in combinations. For it is immaterial whether we think of the line drawn, for example, from the point A to the point B, or from the point B to the point A.

From the five digits 1, 2, 3, 4, and 5, how many products may be formed, using two and only two digits at a time? This is a problem in combinations, for example, the product 3×2 is the same as the product 2×3.

From the five digits 1, 2, 3, 4, and 5, how many two-figured numbers may be formed? This is a problem in permutations; for example, the number 32 is different from the number 23.

Exercises

1. Now many products can be formed from the digits 1, 2, 3, 4, and 5 taking (*a*) two and only two digits to form a product; (*b*) three and only three digits to form a product; (*c*) four and only four digits to form a product; and (*d*) five digits to form a product?

2. How many lines may be drawn through five points in a plane, no three of which are upon one straight line?

3. How many planes are determined by five points in space, no four of which are on one plane?

4. How many points of intersection are determined by five straight lines on a plane, if no three intersect in the same point?

5. How many numbers can be formed from the digits 1, 2, 3, 4, and 5 (*a*) if two and only two digits are used in each number; (*b*) if three and only three digits are used in each number; (*c*) if four and only four digits are used; (*d*) if five and only five digits are used?

98. Formula for ${}_nP_r$. Let a, b, c, d, e, f be permuted two at a time. The first letter may be selected in six ways, *i.e.*, the first letter selected may be the a, the b, the c, the d, the e, or the f. When the first letter is selected, the second letter may be selected in five ways from the remaining five letters. Then, by the fundamental principle, the number of ways two letters may be selected from six letters is 6×5, or 30. Thus, ${}_6P_2 = 6 \times 5 = 30$.

If three letters are placed in each permutation, the first can be selected in six ways, the second in five (6 − 1) ways, and the third in four (6 − 2) ways. Thus, ${}_6P_3 = 6 \times 5 \times 4 = 120$.

If four letters are placed in each permutation, the first can be selected in six ways, the second in five ways, the third in four ways, and the fourth in three ways. Thus, ${}_6P_4 = 6 \times 5 \times 4 \times 3 = 360$.

The student will show, by reasoning similar to the above, that ${}_6P_5 = 6 \times 5 \times 4 \times 3 \times 2 = 720$, and ${}_6P_6 = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$.

If n things are permuted r at a time, the first selection is made from the n things, or it may be made in n ways; the second selection is made from the remaining $n - 1$ things, or it may be made in $n - 1$ ways. Then, by the fundamental principle, the first two things may be selected in $n(n - 1)$ ways. The third selection is made from the remaining $n - 2$ things, or it may be made in $n - 2$ ways. Thus, the first three things may be selected in $n\,(n - 1)(n - 2)$ ways. Continuing in this way the first four things may be selected in $n(n - 1)(n - 2)(n - 3)$ ways, the first five things in $n(n - 1)(n - 2)(n - 3)(n - 4)$ ways, the first six things in $n(n - 1)(n - 2)(n - 3)(n - 4)(n - 5)$ ways, and so on. The r things may be selected in

$$n(n - 1)(n - 2)(n - 3) \cdot \cdot \cdot [n - (r - 2)][n - (r - 1)]$$

ways. Therefore

$${}_nP_r = n(n - 1)(n - 2)(n - 3) \cdot \cdot \cdot \text{ to } r \text{ factors,}$$

or $$\mathbf{{}_nP_r = n(n - 1)(n - 2)(n - 3) \cdot \cdot \cdot (n - r + 1)} \qquad (1)$$

If $r = n$, formula (1) reduces to

$${}_nP_n = n(n - 1)(n - 3) \cdot \cdot \cdot \text{ to } n \text{ factors, or}$$

$${}_nP_n = n(n - 1)(n - 2)(n - 3) \cdot \cdot \cdot 3 \times 2 \times 1$$

$${}_nP_n = \lfloor\underline{n} \qquad (2)$$

where $\lfloor\underline{n}$ (read "factorial n") means $1 \times 2 \times 3 \times 4 \cdot \cdot \cdot (n - 2)(n - 1)n$, or the product of all the positive integral numbers from 1 up to and including n.

The symbol ${}_nP_n$, the number of permutations of n things taken all (n) at a time, will be written P_n.

From formula (1),

$${}_{n-1}P_{r-1} = (n - 1)(n - 2)(n - 3) \cdot \cdot \cdot (n - r + 1)$$

found by replacing n by $n-1$, and r by $r-1$. Then

$$ {}_nP_r = n({}_{n-1}P_{r-1}). \tag{3} $$

By multiplying both numerator and denominator of the right-hand side of formula (1) by $\lfloor \underline{n-r}$, it reduces to

$$ {}_n\mathbf{P}_r = \frac{\lfloor \underline{\mathbf{n}}}{\lfloor \underline{\mathbf{n-r}}}. \tag{4} $$

Exercises

1. Solve exercise 5 of the preceding section by formula (4).

2. How many different signals can be given with seven flags of different colors, by hoisting them in a vertical line, any number at a time?

3. How many different numbers can be formed from the digits 1, 2, 3, 4, 5, 6, if a digit cannot be used more than once in any one number?

4. From twenty men how many nines can be formed, assuming that every man can play any position? By nine is not meant merely the aggregate of nine men, but has reference as well to the positions played.

5. Same as exercise 4, excepting that only two can pitch and that only three can catch.

6. Same as exercise 5, excepting that the pitchers and catchers play no other positions.

99. Formula for ${}_nC_r$. Within each combination of n things taken r at a time there are $\lfloor \underline{r}$ permutations. The number of permutations of n things taken r at a time is equal to the product of the number of combinations of n things taken r at a time by the number of permutations within each combination taking r at a time, or

$$ {}_nP_r = \lfloor \underline{r}\ {}_nC_r, $$

or,

$$ \frac{\lfloor \underline{n}}{\lfloor \underline{n-r}} = \lfloor \underline{r}\ {}_nC_r, $$

or

$$ {}_n\mathbf{C}_r = \frac{\lfloor \underline{\mathbf{n}}}{\lfloor \underline{\mathbf{r}}\ \lfloor \underline{\mathbf{n-r}}}. \tag{1} $$

Every time a combination of r things is selected from n things, a combination of $n - r$ things remains. Thus, the number of combinations of n things taken r at a time is equal to the number of combinations of n things taken $n - r$ at a time. This fact can be shown from formula (1) by replacing r by $n - r$, when it reduces to

$$ {}_nC_{n-r} = \frac{\lfloor n}{\lfloor n-r \rfloor \lfloor n-n+r}} = \frac{\lfloor n}{\lfloor n-r \rfloor \lfloor r}, $$

a formula identical with (1).

Exercises

1. Solve exercises 1, 2, 3, and 4, § **97**, by formula (1).

2. There are twenty points in a plane, no three of which are upon the same straight line, excepting five, which are upon one straight line. How many lines are determined by the points?

100. The Binomial Theorem. The product of the n binomials, $(a_1 + b_1)(a_2 + b_2)(a_3 + b_3) \ldots (a_n + b_n)$, is the sum of the partial products formed by taking, in as many ways as possible, one and only one letter from each of the parentheses.

When the binomials are all equal, the product becomes $(a + b)(a + b) \ldots$ to n factors. By taking the first letter from each of the parentheses the partial product a^n is obtained. By taking a b from one of the parentheses, and an a from the remaining $n - 1$ parentheses, a partial product, $a^{n-1}b$, is obtained. Several partial products of this form may be obtained by selecting the b from different parentheses. In fact, it may be selected in n ways, giving n partial products of the form $a^{n-1}b$. By taking the b from two of the parentheses, and the a from the remaining $n - 2$ parentheses, a partial product, $a^{n-2}b^2$, is obtained. Several partial products of this form may be obtained by selecting the b's from different pairs of parentheses. The number of partial products which may be formed in this way is equal to the number of ways in which two b's can be selected from n b's, or, ${}_nC_2$. In the same way it may be shown that the number of partial products of the form $a^{n-3}b^3$ is ${}_nC_3$; the number of partial products of the form a^{n-4}, b^4 is ${}_nC_4$, etc. Thus,

$$ (a + b)^n \equiv a^n + na^{n-1}b + {}_nC_2a^{n-2}b^2 + {}_nC_3a^{n-3}b^3 + {}_nC_4a^{n-4}b^4 + \ldots + b^n, $$

or

$$(a+b)^n \equiv a^n + na^{n-1}b + \frac{n(n-1)}{\lfloor 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{\lfloor 3}a^{n-3}b^3 + \frac{n(n-1)(n-2)(n-3)}{\lfloor 4}a^{n-4}b^4 + \ldots + b^n.$$

This is known as the **binomial expansion,** or **binomial theorem.** The coefficients of the right-hand side of the equation are called the **binomial coefficients.**

For $n = 2$, $(a+b)^2 = a^2 + 2ab + b^2$.

For $n = 3$, $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

For $n = 4$, $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.

For $n = 5$, $(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$.

$$(2+3x^2)^6 \equiv (2)^6 + 6\,(2)^5(3x^2) + \frac{6\times 5}{\lfloor 2}(2)^4(3x^2)^2 + \frac{6\times 5\times 4}{\lfloor 3}(2)^3(3x^2)^3 + \frac{6\times 5\times 4\times 3}{\lfloor 4}(2)^2(3x^2)^4 + \frac{6\times 5\times 4\times 3\times 2}{\lfloor 5}(2)(3x^2)^5 + \frac{6\times 5\times 4\times 3\times 2\times 1}{\lfloor 6}(3x^2)^6,$$

or

$$(2+3x^2)^6 \equiv 64 + 576x^2 + 2160x^4 + 4320x^6 + 4860x^8 + 2916x^{10} + 729x^{12}.$$

If b is negative, it will be seen that every alternate term of the expansion, beginning with the second, is negative.

Exercises

Expand the following:

1. $(a-b)^4$.
2. $(2+b)^6$.
3. $(2-3x)^5$.
4. $(x^2-3)^4$.
5. $(x^2+3x^2)^5$.
6. $(x^{\frac{1}{2}} + x^{-1})^6$.

101. The Binomial Expansion for Negative and Fractional Exponents. In the preceding section the expansion was given for a binomial with a positive integral exponent. It is shown in higher mathematics that the expansion also holds for fractional

and for negative exponents, *providing that the numerical value of b is less than the numerical value of a.*

For a positive integral exponent the binomial expansion consists of a finite number of terms, $n + 1$. For fractional or negative exponents the number of terms is infinite, or the expansion is said to be an infinite series.

ILLUSTRATION 1: Expand $(1 + x)^{-1}$.

$$(1 + x)^{-1} \equiv (1)^{-1} + (-1)(1)^{-2}(x) + \frac{(-1)(-1-1)}{\underline{|2}}(-1)^{-3}(x)^2 + \frac{(-1)(-1-1)(-1-2)}{\underline{|3}}(-1)^{-4}(x)^3 + \cdots$$

or

$$(1 + x)^{-1} \equiv 1 - x + x^2 - x^3 + \cdots$$

ILLUSTRATION 2: Expand $(1 + x)^{\frac{1}{2}}$.

$$(1 + x)^{\frac{1}{2}} \equiv (1)^{\frac{1}{2}} + (1/2)(1)^{\frac{1}{2}}x + \frac{(1/2)\,(1/2-1)}{\underline{|2}}(1)^{-\frac{3}{2}}(x)^2 + \frac{(1/2)(1/2-1)(1/2-2)}{\underline{|3}}(1)^{-\frac{5}{2}}(x)^3 + \cdots$$

or

$$(1 + x)^{\frac{1}{2}} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \cdots$$

Exercises

Expand the following to four terms:

1. $(1 + x)^{\frac{1}{3}}$.
2. $(1 - x)^{\frac{1}{3}}$.
3. $(1 + x)^{\frac{2}{3}}$.
4. $(1 - x)^{-1}$.
5. $(1 + x)^{-\frac{1}{2}}$.
6. $(1 - x)^{-\frac{1}{3}}$.

102. Approximation Formulas. If x is numerically very small the expansion

$$(1 + x)^n \equiv 1 + nx + \frac{n(n-1)}{\underline{|2}}x^2 + \frac{n(n-1)(n-2)}{\underline{|3}}x^3 + \cdots$$

is approximately equal to

$$1 + nx.$$

For, x^2, x^3, . . ., are higher powers of a small fraction and hence

small compared with x. If the symbol $\doteq$ represents "approximately equal,"

$$(1 + x)^n \doteq 1 + nx \tag{1}$$

if x is numerically very small. Thus, to illustrate,

$$(1 + 0.06)^{\frac{1}{3}} \doteq 1 + \tfrac{1}{3}(0.06) = 1.02.$$

The true value of this expression is, to five decimal places, 1.01961. Again:

$$(1 - 0.06)^{\frac{1}{3}} \doteq 1 - \tfrac{1}{3}(0.06) = 0.98.$$

If x, y and z are numerically very small, the student will show by multiplication and division that the following are approximate formulas:

$$(1 + x)(1 + y) \doteq 1 + x + y, \tag{2}$$

$$\frac{1 + x}{1 + y} \doteq 1 + x - y, \tag{3}$$

$$(1 + x)(1 + y)(1 + z) \doteq 1 + x + y + z. \tag{4}$$

It is to be remembered that the above formulas hold for negative values of x, y, and z as well as for positive values.

Thus, by formula (2)

$$\begin{aligned}(1 + 0.03)(1 - 0.05) &\doteq 1 + 0.03 - 0.05\\ &\doteq 0.98.\end{aligned}$$

Exercises

Find the approximate value of each of the following:

1. $(1.02)^{\frac{1}{2}}$.
2. $(0.98)^{\frac{1}{2}}$.
3. $(0.93)^{\frac{1}{3}}$.
4. $(1.03)(1.02)$.
5. $(1.03)\,(0.98)$.
6. $\dfrac{1.03}{1.02}$.
7. $\dfrac{1.03}{0.97}$.
8. $(1.04)(1.06)(0.95)$.

103. The Compound Interest Law. If a dollars are loaned at r percent per annum, compound interest,

at the end of one year the amount is $a\left[1 + \dfrac{r}{100}\right]$,

at the end of two years the amount is $a\left[1+\frac{r}{100}\right]^2$,

at the end of three years the amount is $a\left[1+\frac{r}{100}\right]^3$,

and at the end of x years the amount is $a\left[1+\frac{r}{100}\right]^x$.

If the interest is compounded semi-annually instead of annually the amount at the end of x years is

$$a\left[1+\frac{r}{200}\right]^{2x},$$

if compounded monthly the amount at the end of the same period is

$$a\left[1+\frac{r}{1200}\right]^{12x},$$

and if compounded n times a year the amount is

$$a\left[1+\frac{r}{100n}\right]^{nx}.$$

Now, if we find the limit of this expression as n increases indefinitely, we find the amount *if the interest were compounded continuously*. For convenience let $\frac{r}{100n}$ be represented by $\frac{1}{u}$. Then the amount, y, is[1]

$$y=\lim_{n\doteq\infty} a\left[1+\frac{1}{u}\right]^{\frac{urx}{100}},$$

which may be written

$$y=a\left[\lim_{u\doteq\infty}\left[1+\frac{1}{u}\right]^u\right]^{\frac{rx}{100}}, \tag{1}$$

since a, r, and x are independent of n. By the binomial theorem,

$$\left[1+\frac{1}{u}\right]^u=1+u\left[\frac{1}{u}\right]+\frac{u(u-1)}{\lfloor 2}\left[\frac{1}{u}\right]^2+\frac{u(u-1)(u-2)}{\lfloor 3}\left[\frac{1}{u}\right]^3+\cdots$$

$$=1+1+\frac{\left[1-\frac{1}{u}\right]}{\lfloor 2}+\frac{\left[1-\frac{1}{u}\right]\left[1-\frac{2}{u}\right]}{\lfloor 3}+\cdots \tag{2}$$

[1] $\lim_{n\doteq\infty} U$ means the limit of U as n increases indefinitely.

Since u becomes ∞ as n becomes ∞, equation (2) gives

$$\lim_{u \doteq \infty} \left[1 + \frac{1}{u}\right]^u = 1 + 1 + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \frac{1}{\underline{|4}} + \frac{1}{\underline{|5}} + \cdots \quad (3)$$

The sum of the infinite series on the right-hand side of (3) is represented by e and is approximately equal to 2.71828. It is the base of the natural or hyperbolic system of logarithms (see footnote, page 66). Equation (1) then reduces to

$$\mathbf{y = ae^{\frac{rx}{100}}}. \quad (4)$$

Since r is any constant, $r/100$ is any constant, call it b. Then equation (4) may be written

$$\mathbf{y = ae^{bx}}, \quad (5)$$

where a and b are constants. It is the equation then which expresses the law: *the rate of growth of a variable, y, is directly proportional to the variable itself.* The multiplier a is the value of the variable when time, x, is zero, and b is the proportionality factor, *i.e.*, it expresses the ratio of the rate of growth of y to y.

The compound interest law is one of the very important laws of nature.

ILLUSTRATION: Suppose it is found by experiment that 10 percent of light is absorbed in passing through a pane of glass 1 cm. thick. Then in passing through a second pane 1 cm. thick, 10 percent of the remaining 90 percent will be lost, and so on. The intensity of light in passing into the glass obeys then the compound interest law,

$$y = ae^{bx}$$

where y represents intensity of light and x distance into the glass. When $x = 0$, $y = a$; thus, a is the intensity of the light as it falls upon the glass surface. When $x = 1$, $y = 0.9a$, hence

$$0.9a = ae^b,$$

or

$$0.9 = e^b$$

or

$$b = \log_e 0.9$$

Then with the constants a and b determined, the intensity of the light at any depth below the surface is given by

$$y = ae^{x \log 0.9} = ae^{\log (0.9)^x} = a(0.9)^x$$

CHAPTER VIII

PROGRESSIONS

104. Arithmetical Progression Defined. An arithmetical progression is a sequence of numbers such that any number of the sequence except the first may be obtained from the preceding by the addition (or subtraction) of a common number, called the **common difference.** The following are arithmetical progressions:

1. 2, 4, 6, 8, 10, 12.	The common difference is 2.
2. 2, 5, 8, 11, 14.	The common difference is 3.
3. $-7, -2, 3, 8, 13$.	The common difference is 5.
4. $-4, -2, 0, 2, 4, 6$.	The common difference is 2.
5. $2\frac{1}{2}, 2, 1\frac{1}{2}, 1, \frac{1}{2}, 0, -\frac{1}{2}$.	The common difference is $-1/2$.

Exercises

From the following sequences select those which are arithmetical progressions, and give the common difference:

1. 5, 7, 9, 11, 13.	**5.** $x - y, x, x + y$.
2. 2, 4, 8, 16.	**6.** $x, xy, x + y$.
3. 10, 8, 6, 4, 2.	**7.** $0, -3, -6, -9$.
4. $3, 2\frac{2}{3}, 2\frac{1}{3}, 2, 1\frac{2}{3}$.	**8.** $a, a + d, a + 2d$.

105. The General Term; the Sum of an Arithmetical Progression. Let a represent the first term, l the last term, d the common difference, n the number of terms, and s the sum of all the terms.

The general series is then

$$a, a + d, a + 2d, a + 3d, \ldots, a + (n - 1)d.$$

It is obvious that the coefficient of d in any term is one less than the number of the term. Thus, in the fourth term it is 3, in the twelfth term it is 11, and in the nth term it is $n - 1$.

A formula for the last term is then

$$\mathbf{l = a + (n - 1)d.} \qquad (1)$$

The sum, s, is found by adding the n terms together, as

$$s = a + (a + d) + (a + 2d) + \cdot \cdot \cdot + [a + (n - 1)d],$$

or

$$s = na + d[1 + 2 + 3 + 4 + \cdot \cdot \cdot + (n - 1)]. \qquad (2)$$

The sum of the $n - 1$ terms written within the brackets may be represented graphically as follows: Let A_0A_{n-1}, Fig. 94, be a line $n - 1$ units long, divided into $n - 1$ equal parts by the points A_1, A_2, A_3, etc. Upon the segments A_0A_1, A_1A_2, A_2A_3, . . ., $A_{n-2}A_{n-1}$, construct rectangles with altitudes respectively 1, 2, 3,

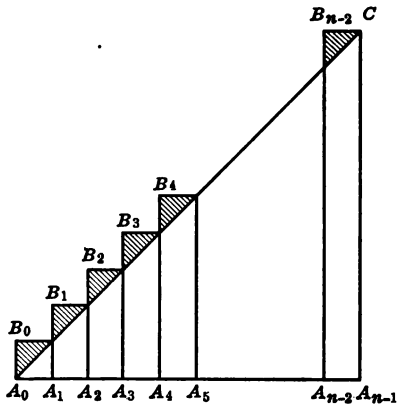


FIG. 94.

4, . . ., $n - 1$ units long. The areas of these rectangles are, respectively, 1, 2, 3, 4, . . ., $n - 1$ square units. The areas of these rectangles may then be taken as representing the terms within the brackets of equation (2), and the sum of the areas of the rectangles representing the sum of the terms within the brackets.

Draw A_0C.

$$A_{n-1}C = A_0A_{n-1}$$

The angle $CA_0A_{n-1} = 45°$

The area of the triangle $A_0A_{n-1}C = \frac{1}{2}(n - 1)^2$

The area of each shaded triangle above the line A_0C is $1/2$.

The sum of the $n-1$ triangles is, then, $\frac{1}{2}(n-1)$. The sum of the $n-1$ rectangles is then

$$\tfrac{1}{2}(n-1)^2 + \tfrac{1}{2}(n-1) = \tfrac{1}{2}(n-1)n.$$

Therefore

$$1 + 2 + 3 + 4 + \cdots + (n-1) = \frac{n(n-1)}{2}.$$

Substituting in (2),

$$s = na + \frac{n(n-1)}{2}d,$$

or

$$\mathbf{s = \frac{n}{2}\left[2a + (n-1)d\right]}. \qquad (3)$$

From formula (3) the sum of an arithmetical progression may be calculated if the first term, the common difference, and the number of terms are given.

Formula (3) may be written.

$$s = \frac{n}{2}\left[a + a + (n-1)d\right],$$

or, since $a + (n-1)d = l$,

$$\mathbf{s = \frac{n}{2}(a + l)}. \qquad (4)$$

ILLUSTRATION 1: Find the sum of 11, 22, 33, . . . to twenty terms.

This is an arithmetical progression, for any term is eleven greater than the next preceding term. $a = 11$, $d = 11$, $n = 20$. Substituting in formula (3),

$$s = \frac{n}{2}\left[2a + (n-1)d\right] = \frac{20}{2}\left[2 \cdot 11 + (20-1)11\right] = 2310.$$

ILLUSTRATION 2: Find the sum of $\sqrt{2} + 2\sqrt{2} + 3\sqrt{2} + \ldots$ to eleven terms.

$$a = \sqrt{2},\ d = \sqrt{2},\ \text{and}\ n = 11.$$

$$s = \frac{11}{2}\left[2\sqrt{2} + (11-1)\sqrt{2}\right] = 66\sqrt{2}.$$

ILLUSTRATION 3: Find the sum of $6 + 5\frac{1}{3} + 4\frac{2}{3} + 4 + \ldots$ to fifty terms.

$$a = 6,\ d = -\tfrac{2}{3},\ \text{and}\ n = 50.$$

$$s = \frac{50}{2}\left[12 + (49)(-\tfrac{2}{3})\right] = 516\tfrac{2}{3}.$$

Exercises

1. Find the sum of $-1-3\frac{1}{2}-6\ \ldots$ to twelve terms.
2. Find the sum of $\frac{1}{5}+\frac{4}{5}+\frac{7}{5}+\ \ldots$ to eighteen terms.
3. Find the sum of the first 100 odd numbers.
4. Find the sum of the first 100 even numbers.
5. A man saves \$200 each year, and at the end of each year places this amount on simple interest at 6 percent. What is the amount of his savings at the end of ten years?
6. For drilling a well a contractor is to receive 10 cents for the first foot, and for each foot thereafter 1/2 cent more than for the preceding foot. What does he receive if the well is 600 feet deep?

106. Geometrical Progression Defined. A geometrical progression is a sequence of numbers such that any number of the sequence, except the first, may be obtained from the preceding term by multiplying by a common number, called the **ratio.** The following are geometrical progressions:

1. $2, 4, 8, 16, \ldots$	The ratio is 2.
2. $2, -4, 8, -16, \ldots$	The ratio is -2.
3. $2, 2\sqrt{2}, 4, 4\sqrt{2}, \ldots$	The ratio is $\sqrt{2}$.
4. $\sqrt{3}, -3, 3\sqrt{3}, -9, \ldots$	The ratio is $-\sqrt{3}$.
5. $a, ar, ar^2, ar^3, \ldots$	The ratio is r.

107. The Last Term; the Sum of a Geometrical Progresssion. Let a represent the first term, l the last term, r the ratio, n the number of terms, and s the sum of the terms of a geometrical progression.

The general sequence is, then, $a, ar, ar^2, ar^3, \ldots, ar^{n-1}$. It is obvious that the exponent of r in any term is one less than the number of the term. Thus, in the fourth term it is 3, in the twelfth term it is 11, and in the nth term it is $n-1$.

A formula for the last term is then

$$l = ar^{n-1}. \tag{1}$$

The sum, s, is found by adding the n terms together, as,

$$s = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1},$$

or

$$s = a(1 + r + r^2 + r^3 + \cdots + r^{n-1}). \tag{2}$$

Since

$$(1 + r + r^2 + r^3 + \cdots + r^{n-1})(1 - r) = 1 - r^n,$$

$$1 + r + r^2 + r^3 + \cdots + r^{n-1} = \frac{1 - r^n}{1 - r},$$

and equation (1) becomes

$$s = \frac{a(1 - r^n)}{1 - r}. \tag{3}$$

ILLUSTRATIONS: 1. Sum the progression $2 + 4 + 8 + \cdots$ to ten terms.

$$a = 2,\ r = 2, \text{ and } n = 10.$$

$$s = \frac{a(1 - r^n)}{1 - r} = \frac{2(1 - 2^{10})}{1 - 2} = 2046.$$

2. Sum the series $\sqrt{2} + 2 + 2\sqrt{2} + \cdots$ to ten terms.

$$a = \sqrt{2},\ r = \sqrt{2}, \text{ and } n = 10.$$

$$s = \frac{\sqrt{2}[1 - (\sqrt{2})^{10}]}{1 - \sqrt{2}} = \frac{31\sqrt{2}}{\sqrt{2} - 1} = 31(2 + \sqrt{2}).$$

3. Sum the series $\sqrt{3} - 3 + 3\sqrt{3} - \cdots$ to ten terms.

$$a = \sqrt{3},\ r = -\sqrt{3}, \text{ and } n = 10.$$

$$s = \frac{\sqrt{3}[1 - (\sqrt{3})^{10}]}{1 - \sqrt{3}} = \frac{242\sqrt{3}}{\sqrt{3} - 1} = 121(3 + \sqrt{3}).$$

Exercises

1. Sum $3 + 9 + 27 + \cdots$ to ten terms.
2. Sum $-3 + 6 - 9 + \cdots$ to ten terms.
3. Sum $3 - 1 + 1/3 - \cdots$ to six terms.
4. Sum $-3 + 6 + 15 + \cdots$ to seven terms.
5. Sum $\sqrt{3} + 2\sqrt{3} + 4\sqrt{3} + \cdots$ to seven terms.
6. Sum $\sqrt{3} + 2\sqrt{3} + 3\sqrt{3} + \cdots$ to seven terms.
7. Sum $2 - \sqrt{2} + 1 - \cdots$ to six terms.
8. Find the eleventh term in each of the above series.

[1] An exception is made when $r = 1$, but in this case $1 + r + r^2 + r^3 + \cdots + r^{n-1} = n$.

9. Show that the only right triangles whose sides are in arithmetical progression are those whose sides are proportional to 3, 4, and 5.

HINT: Let $x - k, x$, and $x + k$ represent the sides. Find k in terms of x, and build up the progression.

10. A debt of \$10,000 is to be paid in ten years. An equal amount is to be paid at the end of each year. Find this amount if the indebtedness draws interest at 6 percent.

11. On a certain twenty-year life insurance policy \$32 premium is paid annually. What do the premiums amount to by the end of the twentieth year, interest compounded annually at $3\frac{1}{2}$ percent?

12. An equal amount of money is deposited at the end of each year for twenty years as a sinking fund to replace a piece of machinery valued at \$10,000. How much must be deposited each year if the deposits draw 4 percent compound interest?

108. The Infinite Geometrical Progression. From the formula for the sum of a geometrical progression, $s = \frac{a(1 - r^n)}{1 - r}$, it is evident that, if r is greater than unity in numerical value, the numerical value of s increases without limit as r increases without limit. For, as n grows very large, r^n grows large without limit, causing s to grow large without limit.

If r, however, is numerically less than unity, r^n grows smaller and smaller as n increases, and approaches zero as n increases without limit. Thus, the quantity within the parentheses approaches 1, or the limit of s is $a/(1 - r)$ as n increases without limit. The sum, s, of a geometrical progression approaches a definite finite limit as n increases without limit, when r is numerically less than unity. This limit, $a/(1 - r)$, is called the **sum of the infinite geometrical progression.**

ILLUSTRATIONS: 1. Find the sum of the infinite progression $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ to ∞.

$a = 1$ and $r = \frac{1}{2}$.

$$s = \frac{a}{1 - r} = \frac{1}{1 - \frac{1}{2}} = 2,$$

2. Find the sum of the infinite progression $\sqrt{3} + 1 + \frac{1}{3}\sqrt{3} - \cdots$ to ∞.

$a = \sqrt{3}$ and $r = 1/\sqrt{3}$.

$$s = \frac{a}{1 - r} = \frac{\sqrt{3}}{1 - 1/\sqrt{3}} = \frac{3}{\sqrt{3} - 1} = \frac{3(\sqrt{3} + 1)}{2}.$$

3. Express the repeating decimal 0.237237237 . . . in fractional form.

The decimal fraction 0.237237237 . . . means $\frac{237}{1000} + \frac{237}{(1000)^2} + \frac{237}{(1000)^3} + \cdot\ \cdot\ \cdot$, an infinite geometrical progression, in which $a = 237/1000$, and $r = 1/1000$. Substituting in the formula for the sum of an infinite geometrical progression,

$$s = \frac{\frac{237}{1000}}{1 - \frac{1}{1000}} = \frac{237}{1000 - 1} = \frac{237}{999}.$$

Exercises

1. Find the sum of $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots$ to ∞.
2. Find the sum of $\sqrt{2} - \frac{1}{2}\sqrt{6} + \frac{1}{2}\sqrt{2} - \ldots$ to ∞.
3. Find the sum of $\frac{1}{\sqrt{2} + \sqrt{3}} + 1 - \frac{1}{\sqrt{2} - \sqrt{3}} + \ldots$ to ∞.
4. Express 0.7261872618 . . . in fractional form.
5. The middle points of the sides of a square, 10 inches on a side, are joined forming a smaller square. The middle points of the sides of this square are joined forming a third square. This process of forming smaller squares is continued indefinitely. Find the sum of the perimeters of the squares.
6. Same as exercise 5, but use an equilateral triangle 10 inches on a side.

CHAPTER IX

[PROBABILITY]

109. Probability Defined. When a coin is thrown upon the flat surface of a table, it falls in one of two ways, heads or tails. If it is given a whirling motion when thrown, the chances of its falling heads are no greater than the chances of its falling tails, and the chances of its falling tails are no greater than the chances of its falling heads; that is, before the coin fell, we had no reason for expecting one rather than the other. All that could be said was that the coin must fall in one of two ways, both *equally likely.*

When a common die is thrown it falls in one of six possible ways, all *equally likely.* One and only one of the six ways is that of the ace up. The chances then of throwing an ace with one throw of a die is one in favor and five against, or one to five, or one out of six. This same thought is expressed by saying that the **probability** of throwing an ace with one throw of a die is one to six, or 1/6.

If a thing can happen in n ways, and can fail to happen in m ways, it can happen or fail in just $m + n$ ways. We say there **are m + n possible cases, m favorable cases, and n unfavorable cases.** If all these ways are equally likely to occur, $\frac{m}{m+n}$ (the ratio of the favorable cases to the possible cases) is called the **probability** of the event happening. $\frac{n}{m+n}$ (the ratio of the unfavorable cases to the possible cases) is called the **probability** of it failing.

When a coin is thrown, the probability of heads turning is 1/2; for the number of favorable cases is one, and the number of possible cases is two. The probability of heads not turning is 1/2.

When a die is thrown, the number of possible cases is six, the number of favorable cases of the ace falling up is one. The probability of throwing an ace is then 1/6. The probability of not throwing an ace is 5/6.

set of twenty-eight dominoes there are seven *doublets.*
ıne domino is selected at random the probability *of its*
, doublet is 7/28, or 1/4. For, the selection *of one*
may be made is twenty-eight ways, or the number *of*
cases is twenty-eight; and since a doublet may be
in seven ways, the number of favorable cases is seven.
ı two dice are thrown the number of ways in which they
l is $6 \times 6 = 36$. Of these thirty-six ways, one, and one
a pair of aces. The probability of throwing a pair of aces
o dice is then 1/36. The probability of throwing a pair
or 1/6. The probability of throwing a sum less than

event is certain to happen, the number of favorable cases
to the number of possible cases, and the probability of
ıening is one. If an event is certain to fail the number of
le cases is zero, and the probability of its happening is
n all other cases the probability of an event happening
tive proper fraction.
event may happen in **a** ways, and fail in **b** ways, the
ity of its happening is $\frac{a}{a+b}$, and the probability of its
ı $\frac{b}{a+b}$. The sum of these two probabilities is **1.**

Exercises

o dice are thrown. What is the probability of throwing a
ı? A sum greater than 7? A sum equal to 10? Conse-
ımbers? One and only one 6?
ox contains three white, two black, and five red balls; if one
, what is the probability it is white? Black? Red? If two
n, what is the probability that both are white? Both black?
? One white and one black? One white and one red?
k and one red?
ee coins are thrown. What is the probability that all turn
That two turn heads? That only one turns heads?
a box there are five red, two white, and seven blue balls.
are drawn one after the other, what is the probability of
ıg drawn in the order red, white, and blue?

HINT: The total number of ways, in order, in which three balls can be drawn from fourteen balls is ${}_{14}P_3$.

5. Four cards are drawn from a pack. What is the probability that there is one from each suit?

110. Mutually Exclusive Events. Two or more events are said to be mutually exclusive if the occurrence of one excludes the occurrence of the others. Thus, a 5 and a 6 cannot be thrown together with one die. The occurrence of one event excludes the occurrence of the other. The two events are mutually exclusive.

Let A and B be two mutually exclusive events. There are three possible cases:

(1) The event A may happen and the event B fail.
(2) The event B may happen and the event A fail.
(3) The event A and the event B may both fail.

If a, b, and c are, respectively, the number of equally likely possible cases, the probability that either A or B happens is $\frac{a+b}{a+b+c}$. For $a+b+c$ is the number of possible cases, and $a+b$ is the number of favorable cases. The probability that A happens is $\frac{a}{a+b+c}$, and that B happens is $\frac{b}{a+b+c}$. Hence *the probability of the occurrence of one or the other of two mutually exclusive events is the sum of the probabilities of the two separate events.*

It is easily seen that this law may be extended to *any number* of mutually exclusive events. The above law applies only to mutually exclusive events, and care must be exercised not to apply it to events not so related. Thus, if A's probability of winning a contest is 1/3, and if B's probability of winning in the same contest is 1/4, the probability of A or B winning is $\frac{1}{3}+\frac{1}{4}=\frac{7}{12}$. For the two events are mutually exclusive. If A's probability of solving a problem is 1/3, and B's probability of solving the same problem is 1/4, the probability of the problem being solved is not the sum of the separate probabilities, for the events are not mutually exclusive. The solution of the problem by A does not prevent B from solving it.

Exercises

1. The probability of throwing an ace with a common die is $\frac{1}{6}$; the probability of throwing a 6 is $\frac{1}{6}$. What is the probability of throwing an ace or a 6?

2. A ticket is drawn from a set numbered from 1 to 50. What is the probability that it is a multiple of either 7 or 9?

3. A die is thrown. What is the probability that the throw will be greater than 3?

4. Two dice are thrown. What is the probability that the throw will be greater then 7?

5. A bag contains four red, five white, two blue, and seven green balls. One is drawn. What is the probability that it is either red or blue?

111. Independent Events. The occurrence of several single events simultaneously or in succession constitutes a compound event. Events are said to be **dependent** or **independent,** if the occurrence of one does or does not affect the occurrence of the others.

Thus, the drawing of a red and a white ball from a bag containing red, white, and blue balls is a compound event.

Of two independent events, let a be the number of ways, all equally likely, the first may happen, and let b be the number of ways, all equally likely, it may fail; let a_1 be the number of ways, all equally likely, the second may happen, and let b_1 be the number of ways, all equally likely, it may fail. The number of possible cases for the first event is $a + b$, and the number of possible cases for the second event is $a_1 + b_1$. Each case of the first event may be associated with each case of the second event. The number of possible cases of the compound event is then $(a + b)(a_1 + b_1)$, all equally likely to occur. In aa_1 cases both events happen; in bb_1 cases both events fail; in ab_1 cases the first event happens and the second event fails; and in a_1b cases the first event fails and the second event happens. Then the probability

$$\text{of both events happening is } \frac{aa_1}{(a + b)(a_1 + b_1)},$$

$$\text{of both events failing is } \frac{bb_1}{(a + b)(a_1 + b_1)},$$

of the first happening and second failing is $\frac{ab_1}{(a+b)(a_1+b_1)}$,

of the first failing and second happening is $\frac{a_1b}{(a+b)(a_1+b_1)}$.

From the first probability given above it is seen that *the probability of the occurrence of two independent events is the product of the probabilities of the occurrence of the separate events.*

This law may be extended to include three or more single events.

ILLUSTRATIONS: 1. A bag contains three white and four black balls. The probability of drawing a white ball is 3/7. If a white ball is drawn and not replaced, the probability of drawing another white ball is 2/6. The probability of drawing two white balls in succession is $\frac{3}{7} \times \frac{1}{3} = \frac{1}{7}$.

2. If A's probability of solving a problem is 1/3, and if B's probability of solving it is 1/4, the probability that A does not solve it is 2/3, and the probability that B does not solve it is 3/4. The probability that both fail to solve the problem is $\frac{2}{3} \times \frac{3}{4} = \frac{1}{2}$. Since 1/2 is the probability the problem is not solved, 1 − 1/2, or 1/2, is the probability that the problem will be solved by A or B.

3. Five coins are thrown. What is the probability that they will all turn heads? The probability that a coin turns heads is 1/2. Hence the probability that the five turn heads is $\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{32}$.

Exercises

1. What is the probability that at least one ace will turn when three dice are thrown?

2. A bag contains six white and five red balls, and a second bag contains four white and three black balls. A ball is drawn from each, what is the probability that both are white? That one is white and one black? That one is white and one red?

3. What is the probability of holding four aces in a game of whist?

4. A sack contains three white and two black balls; a second sack contains four white and seven black balls. A ball is drawn from the first sack and placed in the second; then a ball is drawn from the

second sack. What is the probability that the second ball drawn is white?

5. A bag contains three white and four black balls. Two persons draw alternately one ball without replacing it. If *A* draws first, what is the probability that he is the first to draw a white ball?

112. Expectation. If p denotes the probability of a person winning a sum of money, m, the product mp is called the value of his expectation, or his mathematical expectation. Thus, if a man is to receive \$10 if he draw a doublet from a set of twenty-eight dominoes with one trial, his mathematical expectation is \$2.50, found by multiplying \$10 by 1/4, the probability of drawing a doublet.

113. Successive Trials. On a single trial, let p be the probability of an event happening and let q be the probability of the event not happening. Thus, if a die is thrown but once, the probability of throwing an ace is $1/6 = p$, and the probability of not throwing an ace is $5/6 = q$. If the die is thrown twice, the probability of throwing an ace each time is p^2, or $(1/6)^2 = 1/36$. The probability of not throwing an ace is $q^2 = 25/36$. The probability of throwing an ace on the first trial and not throwing an ace on the second trial is pq, or $\frac{1}{6} \times \frac{5}{6} = \frac{5}{36}$. The probability of not throwing an ace on the first trial and throwing an ace on the second trial is qp, or $\frac{5}{6} \times \frac{1}{6} = \frac{5}{36}$. The probability of throwing an ace once and once only with two trials is $\frac{5}{36} + \frac{5}{36} = \frac{10}{36}$. The probability of throwing an ace at least once with two trials is $\frac{10}{36} + \frac{1}{36} = \frac{11}{36}$.

The probability of the event happening on each of the two trials is p^2. The probability of the event happening on the first trial and not happening on the second trial is pq. The probability of the event not happening on the first trial and happening on the second trial is qp. The probability of the event happening but once is $2pq$. The probability of the event not happening upon either of the trials is q^2.

$$p^2 + 2pq + q^2 = 1$$

In the illustration above,

$$p^2 + 2pq + q^2 = \frac{1}{36} + \frac{10}{36} + \frac{25}{36} = \frac{36}{36} = 1$$

The probability of the event happening on each of three trials is p^3. The probability of it happening on the first two trials and not happening on the last trial is p^2q; of happening on the first and last trials and not happening on the second trial is p^2q; and of happening on the last two trials and not happening on the first trial is p^2q. The probability of the event happening twice and twice only with three trials is $3p^2q$. The probability of the event happening once and once only with three trials is $3pq^2$. The probability of the event failing to happen with three trials is q^3. The probability that the event happens at least once with three trials is $p^3 + 3p^2q + 3pq^2$.

$$p^3 + 3p^2q + 3pq^2 + q^3 = 1$$

A sack contains three white balls and seven black balls. Three successive draws of one ball are made, each ball being returned to the sack before the next draw is made. $p = 3/10$ and $q = 7/10$. The probability of drawing a white ball each time is $(3/10)^3 = 27/1000$; of drawing two white balls and one black ball is $3(3/10)^2(7/10) = 189/1000$; of drawing one white ball and two black balls is $3(3/10)(7/10)^2 = 441/1000$; and of drawing a black ball each time is $(7/10)^3 = 343/1000$. The probability of drawing a white ball at least once is

$$\frac{27}{1000} + \frac{189}{1000} + \frac{441}{1000} = \frac{657}{1000}.$$

If n trials are made the probability of the event happening on each trial is $\mathbf{p^n}$. The probability of the event happening $n - 1$ times and failing to happen one time is $\mathbf{np^{n-1}q}$. For, the probability of its failing to happen on the last trial only is $p^{n-1}q$, of its failing to happen on the next to the last trial only is $p^{n-1}q$, of its failing to happen on the second from the last trial only is $p^{n-1}q$, and so on for the n trials on which the event may fail to happen; then the probability of the event failing to happen one time only in the n trials is the sum of these probabilities, or $\mathbf{np^{n-1}q}$. In a similar way it is shown that the probability

of the event happening on $n-2$ trials and failing to happen on two trials is ${}_nC_2p^{n-2}q^2$ (where ${}_nC_2$ is the number of combinations of n things taken two at a time), or $\frac{\lfloor n}{\lfloor 2 \mid \lfloor n-2}\, p^{n-2}q^2$. The probability that the event happens on $n-3$ trials and fails to happen on three trials is $\frac{\lfloor n}{\lfloor 3 \mid \lfloor n-3}\, p^{n-3}q^3$. The probability that the event happens on $n-r$ ($r \lesseqgtr n$) trials and fails to happen on r trials is $\frac{\lfloor n}{\lfloor r \mid \lfloor n-r}\, p^{n-r}q^r$. The probability that the event happens on each trial, on each trial but one, on each trial but two, on each trial but three, etc., are, respectively, the first, second, third, etc., terms of the binomial expansion,

$$(p+q)^n = p^n + np^{n-1}q + \frac{\lfloor n}{\lfloor 2 \mid \lfloor n-2}\, p^{n-2}q^2 + \frac{\lfloor n}{\lfloor 3 \mid \lfloor n-3}\, p^{n-3}q^3$$
$$+ \cdots + \frac{\lfloor n}{\lfloor r \mid \lfloor n-r}\, p^{n-r}q^r + \cdots + q^n = 1 \quad .$$

The probability that the event happens on k trials and on k trials only is $\frac{\lfloor n}{\lfloor k \mid \lfloor n-k}\, p^kq^{n-k}$. The probability that the event happens on at least k trials out of n trials is

$$p^n + np^{n-1}q + \frac{\lfloor n}{\lfloor 2 \mid \lfloor n-2}\, p^{n-2}q^2 + \cdots$$
$$+ \frac{\lfloor n}{\lfloor n-k-1 \mid \lfloor k+1}\, p^{k+1}q^{n-k-1} + \frac{\lfloor n}{\lfloor n-k \mid \lfloor k}\, p^kq^{n-k}.$$

Thus, if a die is thrown six times, the probability of throwing an ace three times and three times only is

$$\frac{\lfloor 6}{\lfloor 3 \mid \lfloor 6-3}(1/6)^3(5/6)^3 = \frac{10{,}000}{186{,}624}.$$

Exercises

1. What is the probability of throwing exactly three 6's in five throws with a single die?

2. The odds are 2:1 in favor of A winning a single game against B. Find the probability of A winning at least two games out of three.

[114.] Graphic Representation of the Terms of $(p + q)^n$. Let A and B represent, respectively, the $(r-1)$st and rth terms of the expansion of $(p+q)^{n-1}$, and let C represent the rth term of the expansion of $(p+q)^n$. Then:

$$A = {}_{n-1}C_{r-1}p^{n-r+1}q^{r-2} = \frac{\lfloor n-1}{\lfloor r-2 \ \lfloor n-r+1} p^{n-r+1}q^{r-2},$$

$$B = {}_{n-1}C_{r-1}p^{n-r}q^{r-1} = \frac{\lfloor n-1}{\lfloor r-1 \ \lfloor n-r} p^{n-r}q^{r-1},$$

$$C = {}_{n}C_{r-1}p^{n-r+1}q^{r-1} = \frac{\lfloor n}{\lfloor r-1 \ \lfloor n-r+1} p^{n-r+1}q^{r-1}.$$

From these expressions it will be shown that

$$C = Aq + Bp.$$

$$Aq = \frac{\lfloor n-1}{\lfloor r-2 \ \lfloor n-r+1} p^{n-r+1}q^{r-1},$$

$$Bp = \frac{\lfloor n-1}{\lfloor r-1 \ \lfloor n-r} p^{n-r+1}q^{r-1},$$

and

$$Aq + Bp = \lfloor n-1 \left[\frac{1}{\lfloor r-2 \ \lfloor n-r+1} + \frac{1}{\lfloor r-1 \ \lfloor n-r} \right] p^{n-r+1}q^{r-1},$$

or

$$Ap + Bq = \lfloor n-1 \left[\frac{r-1}{\lfloor r-1 \ \lfloor n-r+1} + \frac{n-r+1}{\lfloor r-1 \ \lfloor n-r+1} \right] p^{n-r+1}q^{r-1},$$

or

$$Ap + Bq = \frac{\lfloor n}{\lfloor r-1 \ \lfloor n-r+1} p^{n-r+1}q^{r-1} = C.$$

The general term, the rth term, of the expansion of $(p + q)^n$ may be found from the $(r - 1)$st and rth terms of the expansion of $(p + q)^{n-1}$.

This property of the binomial expansion makes it possible to construct graphically the terms of the expansion for the case

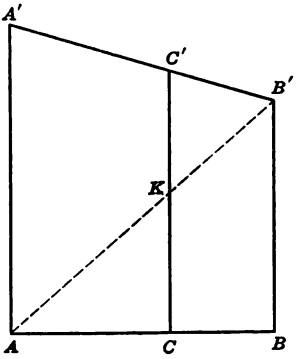


FIG. 95.

$p + q = 1$. Let AA' and BB', Fig. 95, whose lengths represent, respectively, A and B, be drawn perpendicular to AB. CC' is a third perpendicular drawn such that

$$\frac{AC}{p} = \frac{CB}{q}.$$

Draw AB' cutting CC' at K. From the similar triangles $AA'B'$ and $KC'B'$,

$$\frac{KC'}{CB} = \frac{AA'}{AB},$$

or

$$KC' = \frac{AA' \times CB}{AB}. \tag{1}$$

From the similar triangles $BB'A$ and CKA,

$$\frac{CK}{AC} = \frac{BB'}{AB},$$

or

$$CK = \frac{BB' \times AC}{AB}. \tag{2}$$

By adding (1) and (2),

$$KC' + CK = \frac{BB' \times AC + AA' \times CB}{AB},$$

or

$$CC' = \frac{B(AC) + A(CB)}{AB} = \frac{B(AC) + A(CB)}{AC + CB},$$

or

$$CC' = \frac{B(CB)\frac{p}{q} + A(CB)}{(CB)\frac{p}{q} + (CB)} = \frac{Bp + Aq}{p + q},$$

or, since $p + q = 1$, $CC' = Bp + Aq$. Therefore $CC' = C$, the rth term of the expansion $(p + q)^n$, when $p + q = 1$.

The first term of the expansion of $(p + q)^n$ equals the first term of the expansion of $(p + q)^{n-1}$ multiplied by p, and may be found graphically as in Fig. 95 by taking AA' zero. The last term of the expansion of $(p + q)^n$ equals the last term of the expansion of $(p + q)^{n-1}$ multiplied by q, and may be found graphically as in Fig. 95 by taking BB' zero.

Let, 0, 1, 2, 3, 4, 5, 6, 7, . . . , Fig. 96, be points chosen equidistant along a horizontal line. Let C_1, C_2, C_3, C_4, . . . be intermediate points chosen such that

$$\frac{p}{q} = \frac{0C_1}{C_1 1} = \frac{1C_2}{C_2 2} = \frac{2C_3}{C_3 3} = \text{etc.}$$

At these points of division of the line 07 erect perpendiculars and let these perpendiculars be designated by the letter or numeral at its lower end. We shall show how to lay off, by means of a graphical construction, the successive terms of the expansion of $(p + q)^n$. With some convenient scale let $1P = p$ and $2Q = q$. Then $1P$ and $2Q$ represent the terms of the expansion when $n = 1$. Draw OP, PQ, and $Q3$. OP cuts C_1 at α_1, PQ cuts C_2 at β_1, and Q_3 cuts C_3 at γ_1. α_1 projected upon 1 gives a_1, β_1 projected upon

2 gives b_1, and γ_1 projected upon 3 gives c_1. By what was shown in the first part of this section it will be seen that $1a_1$, $2b_1$, and $3c_1$ represent the terms of $(p + q)^2$. In a similar manner it is shown that $1a_2$, $2b_2$, $3c_2$, and $4d_2$ represent the terms of the expansion $(p + q)^3$; $1a_3$, $2b_3$, $3c_3$, $4d_3$, and $5e_3$ represent the terms of $(p + q)^4$; $1a_4$, $2b_4$, $3c_4$, $4d_4$, $5e_4$, and $6m_4$ represent the terms of $(p + q)^5$. Continuing in this way the terms of the expansion could be represented for any positive integral value of n.

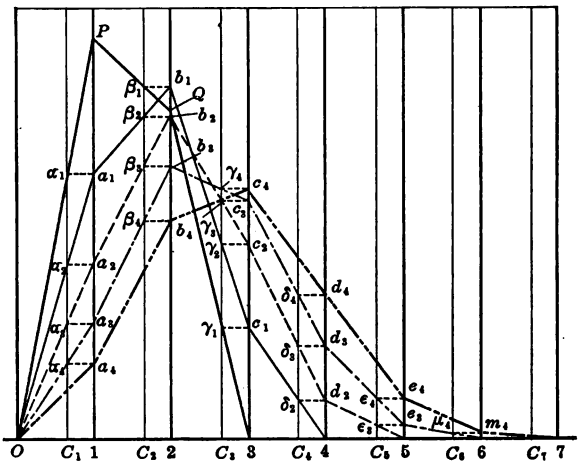


FIG. 96.

If $p + q$ is not 1, but k, the expression $(p + q)^n$ could be written $(kp' + kq')^n$, or $k^n(p' + q')^n$, where $p' + q' = 1$. The terms of $(p' + q')^n$ may now be represented as above and finally all ordinates increased in the ratio of $1 : k^n$, or the vertical scale changed so that all vertical distances represent quantities k^n times as great as originally.

If $p = q$, the configuration, Fig. 96, will be symmetrical with respect to some vertical line. In this case ($p = q = 1/2$) the ordinates represent the binomial coefficients. When $p \neq q$ the configuration is said to be **asymmetrical** or **skew.**

In Fig. 97 the ordinates represent the binomial coefficients of the 999th power of $(p + q)$. The drawing is due to Quetelet.[1]

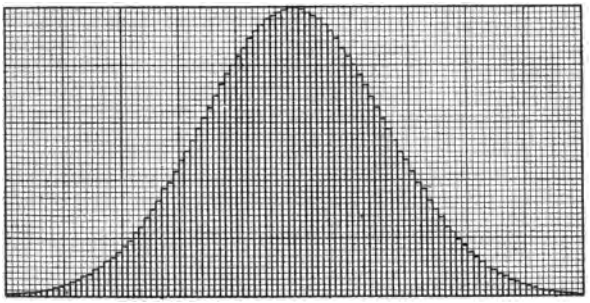


FIG. 97.

From what was shown in the first part of this section the binomial coefficients of any power may be calculated from those of the expansion of a power one lower.

n	Binomial Coefficients							
1	1	1						
2	1	2	1					
3	1	3	3	1				
4	1	4	6	4	1			
5	1	5	10	10	5	1		
6	1	6	15	20	15	6	1	
7	1	7	21	35	35	21	7	1

Thus, in the above, any number is the sum of the number directly above it and the number to the left of the latter.

Exercises

1. Represent graphically, as in Fig. 96, the terms of $(p + q)^n$, if $p = \frac{2}{3}$, $q = \frac{1}{3}$, and $n = 7$.

[1] The limit of the broken line at the top of the ordinates is known as the probability curve; its equation is of the form

$$y = ae^{-b^2x^2}$$

as is shown in treatises on the Theory of Probability.

2. Represent graphically, as in Fig. 96, the terms of $(p + q)^n$, if $p = q = \frac{1}{2}$ and $n = 7$.

115. Deviations from the Arithmetic Mean and the Probability Curve. The readings of a set of direct measurements of a magnitude (as the elevation of a bench mark, the length of a base line, the angle subtended by a distant object, etc.) taken by the same person or by several persons, will, in general, disagree among themselves. This is especially true if the measuring instrument is such as to enable the observer to read to a small part of the unit; for example, as the reading of a level rod to the nearest one-thousandth of a foot.

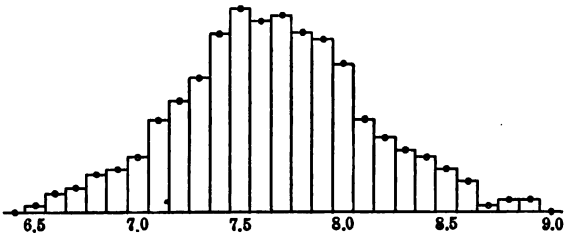


FIG. 98.—A frequency distribution polygon.

It is assumed that the measuring instruments are in adjustment and that proper care is exercised in handling and reading them. The disagreements of the readings are then due to what are known as **errors of observation.**

In Table XIV are given readings of the area bounded by an irregular curve drawn at random. The readings were taken with an Amsler Polar Planimeter. In column I are given the readings of the instrument and in column II are given the number of times the corresponding reading was obtained. Fig. 98 gives a graphic representation of the distribution of the frequency of the readings. Areas are represented by abscissas and frequencies of readings by ordinates. To represent more clearly to the eye the distribution of the frequencies of the readings, the plotted points may be joined, in order, by straight lines; or through each point a horizontal line may be drawn giving a series of rectangles, as is illustrated by the figure. The length of the base of each rectangle,

in this case, represents the least count of the instrument, while its altitude represents the frequency of a reading.

TABLE XIV

I.	II.	III	IV	V	VI
Area, sq. cm.	No. of readings	f	v	fv^2	
126.4	0	1	−12.6	158.76	.....
126.5	1	4	−11.6	538.24	.....
126.6	3	8	−10.6	898.88	8
126.7	4	13	− 9.6	1198.08	.....
126.8	6	17	− 8.6	1257.32	.....
126.9	7	22	− 7.6	1270.72	22
127.0	9	31	− 6.6	1350.36	.....
127.1	15	42	− 5.6	1317.12	.....
127.2	18	55	− 4.6	1163.80	55
127.3	22	69	− 3.6	894.24	.....
127.4	29	84	− 2.6	567.84	.....
127.5	33	93	− 1.6	238.08	93
127.6	31	96	− 0.6	35.56	.....
127.7	32	92	0.4	14.72	.....
127.8	29	89	1.4	150.41	89
127.9	28	81	2.4	465.56	.....
128.0	24	67	3.4	774.52	.....
128.1	15	51	4.4	987.36	51
128.2	12	37	5.4	1078.92	.....
128.3	10	31	6.4	1269.76	.....
128.4	9	26	7.4	1423.76	26
128.5	7	21	8.4	1481.76	.....
128.6	5	13	9.4	1148.68	.....
128.7	1	8	10.4	865.28	8
128.8	2	5	11.4	649.80	.....
128.9	2	4	12.4	615.04	.....
129.0	0	2	13.4	359.12	2
Sums ..	354	1062		22,173.68	

Mean area = 127.66 sq. cm. $\qquad \sigma = 4.3$

$h^2 = 0.024 \qquad E_{s.r} = 3.08$ or 0.308 sq. cm.

$\frac{h}{\sqrt{\pi}} = 0.087 \qquad E = 0.1$ or 0.01 sq. cm.

The graph obtained by either method of connecting the plotted points is called **a frequency distribution polygon.**

From Fig. 98, as well as from the table of readings, one is lead to believe that the true area sought lies somewhere between 127.3 and 128.0, probably near 127.6 square centimeters.

If it be granted that *in the long run* numerically small errors occur more frequently than larger ones, and that positive and negative errors, numerically equal, occur with equal frequency, the frequency polygon for a *very large number of readings* would present symmetry with respect to a vertical line, and the altitudes of the rectangles would decrease with increasing distances from this line of symmetry. In such a case the altitude of any rectangle, or its area, would be proportional to the probability that a single reading fall within the limits defined by the ends of its base. For, the sum of the altitudes of all rectangles would represent the total, or possible, number of cases, while the altitude of a particular rectangle would represent the number of cases of the corresponding reading. Since the bases of the rectangles are equal, the above statement holds for the areas of the rectangles as well as for their altitudes.

If the scales of the drawing are so chosen that the sum of the areas of the rectangles is unity, the sum of the areas of any number of adjacent rectangles is equal to the probability that a single reading falls within the limits represented by the extremes of the bases of the rectangles.

In a large number of cases it has been found that the plotted points representing the frequency distribution of direct observations follow the curve representing the equation

$$y = ae^{-h^2x^2} \tag{1}$$

where e is the base of the hyperbolic logarithmic system, approximately equal to 2.7183, a and h are positive constants, and x the variable distance from the vertical line of symmetry, *i.e.*, distance from the true value. Thus, x represents the error of a reading. It is also found that the line of symmetry corresponds to the arithmetic mean of the readings. (See Figs. 99 and 100. Also Figs. in Methods of Least Squares, by G. C. Comstock.)

If the area under the curve is taken as unity, equation (1) becomes

$$y = \frac{h}{\sqrt{\pi}} e^{-h^2x^2} \tag{2}$$

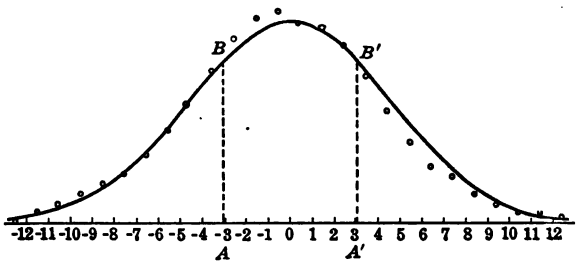


FIG. 99.—Graphic representation of frequency distribution of readings given in Table XIV. The circles represent the distribution of the readings, while the curve is the probability curve calculated from them. The equation of the curve is $y = 0.087e^{-0.024x^2}$. The probable error of the mean is 0.01 square centimeters.

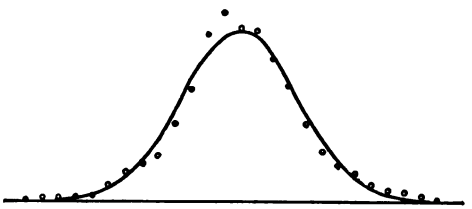


FIG. 100.—Graphic representation of frequency distribution of readings of the elevation of a bench mark. The circles represent the distribution of the readings, while the curve is the probability curve calculated from them.

In what follows it is assumed that the units are chosen such that the area under the curve is unity, so that the number of units of area bounded by the curve (2), the X-axis, and any two vertical lines is the probability that a single reading falls within the limits corresponding to the positions of the vertical lines, or

the probability that a single reading has an error falling within the values of x defining the vertical lines.[1]

The curve represented by equation (2) is known by various names, as **"the probability curve," "the error curve," "the normal distribution curve."**

While, probably, no set of readings which the student will take, or will have occasion to study, will be large enough in number to give a frequency distribution polygon following very closely the probability curve, yet the laws and formulas derived from this law will be used and applied to the data at hand. Thus, to illustrate, the empirical law states that for a very, very large number of observations the line of symmetry corresponds to the arithmetic mean of the readings, and hence this mean gives the

[1] For students who have a knowledge of the Calculus, and who may chance to read this chapter, a few proofs of statements, which must of necessity be taken for granted by the students for whom this book is intended, are inserted in footnotes.

The curve for

$$y = ae^{-h^2x^2}$$

is symmetrical with respect to the Y-axis and approaches the X-axis asymptotically. y has a maximum value when $x = 0$. If the area under the curve is unity, it follows that

$$a\int_0^\infty e^{-h^2x^2}dx = \tfrac{1}{2},$$

or, if $h^2x^2 = t$,

$$\frac{a}{2h}\int_0^\infty t^{\frac{1}{2}-1}e^{-t}dt = \tfrac{1}{2},$$

or

$$\frac{a}{2h}\Gamma(\tfrac{1}{2}) = \tfrac{1}{2},$$

or

$$\frac{a}{2h}\sqrt{\pi} = \tfrac{1}{2},$$

or

$$a = \frac{h}{\sqrt{\pi}}.$$

Hence

$$y = \frac{h}{\sqrt{\pi}}e^{h^2x^2}$$

The probability that a single reading will have an error between α and β is

$$\frac{h}{\sqrt{\pi}}\int_\alpha^\beta e^{-h^2x^2}dx$$

true value of the measured magnitude. But for a very limited number of readings the arithmetic mean gives, probably, only a very close approximation to the true value. The empirical law, however, does seem to indicate that for a set of repeated readings the arithmetic mean is the best mean to use to represent the true value.

116. Smoothing the Distributing Polygon. If only a few readings are taken the frequency distribution polygon will in general be very irregular. A more regular polygon may often be obtained by grouping the readings within some larger interval. This method is illustrated in column III, of Table XIV, and by Fig. 101. In column

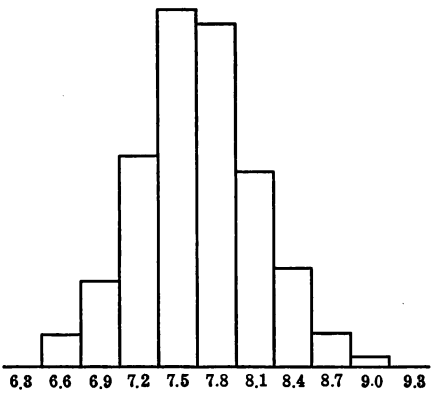


FIG. 101.—Frequency distribution polygon.

III, to illustrate, 93 corresponding to area 127.5 is the sum of the frequencies 29, 33 and 31 of column II, corresponding, respectively, to the areas 127.4, 127.5 and 127.6. In Fig. 101 the numbers of column III are plotted against areas.

Another method of smoothing the curve is to consider that in a larger number of readings the frequency of any one would be proportional to the mean of the observed frequencies of it and its two adjacent readings. The frequencies given in column III of Table XIV are obtained in this way from the observed frequen-

cies given in column II.[1] In Fig. 99 frequencies, f, are plotted against areas.

117. Measure of Precision; Standard Deviation. For a set of readings it may not be enough to know merely the value of the arithmetical mean. One may desire to know what degree of confidence may be placed in this mean, or to know something of the agreement or disagreement among the individual readings. This information is given by the table of readings or by the frequency distribution polygon, but it is desirable to have some definite mathematical expression giving the precision of the set of readings.

If in a set of readings the number of small errors is relatively large compared with the number of small errors of a second set of readings, the first set of readings would, in general, command more confidence than the second set. Of the two corresponding frequency distribution polygons the first would be tall and narrow as compared with the second. Since the probability curve for the first set of readings will have a greater maximum ordinate than that for the second set, and since $\frac{h}{\sqrt{\pi}}$ is the value of the maximum ordinate, h will be larger for the first set than for the second. h is called the **measure of precision.** Hence of several sets of readings, that set is considered the best for which h is a maximum.

Let x represent the distance of the center of the base of a rectangle in a frequency distribution polygon from the point corresponding to the arithmetic mean. In Table XIV, these values are given in column IV. It can then be shown that[2]

$$\frac{1}{2h^2} = \Sigma \frac{fv^2}{n} \tag{1}$$

[1] In this case, instead of taking the mean, the sum of the three readings is taken. This is done to avoid the division by 3, and at the same time to avoid fractions. It will be noticed that by this process of smoothing, the arithmetic mean is not changed.

[2]
$$\Sigma y v^2 \Delta v = \frac{2h}{\sqrt{\pi}} \int_0^\infty v^2 e^{-h^2 v^2} dv \quad \text{(approximately)}$$
$$= \frac{1}{h^2\sqrt{\pi}} \int_0^\infty t^{\frac{3}{2}-1} e^{-t} dt, \quad \text{where } h^2 v^2 = t$$
$$= \frac{1}{h^2\sqrt{\pi}} \Gamma(\tfrac{3}{2}) = \frac{1}{2h^2}.$$

$y\Delta x$ is the area of a frequency rectangle (see next footnote). If the horizontal

(where n represents the total number of readings) *provided that the unit of v is the interval corresponding to the base of the frequency rectangle.*[1]

In statistical work $\sqrt{\frac{\Sigma fv^2}{n}}$ is usually denoted by σ (sigma) and is called the **standard deviation.** Equation (1) may then be written

$$\frac{1}{2h^2} = \sigma^2 \tag{2}$$

and the equation of the probability curve may be written:

$$y = \frac{1}{2\sqrt{2\pi}}\, e^{-\frac{x^2}{2\sigma^2}} \tag{3}$$

In practice, instead of using the value of h as a basis of comparison of the precision of sets of readings, the value of σ is used. Since σ varies inversely as h, that set of readings is considered the best which gives σ the smallest value.[2]

scale is chosen so that Δv is unity, the sum of the frequency rectangles will be unity if $y = f/n$, where n is the total number of readings. Thus:

$$\frac{1}{2h^2} = \Sigma\frac{fx^2}{n}.$$

[1] Frequency rectangle, as here used, means a rectangle of the frequency distribution polygon.

[2] From equation (1) it is seen that for h to have a maximum value Σfv^2 must have a minimum value. Σfv^2 is the sum of the squares of the deviations of the readings from the arithmetic mean, the approximate value, or Σfv^2 is the sum of the squares of the residuals as they are called. Thus the sum of the squares of the residuals must be a minimum. This principle is sometimes used in solving simultaneous observation equations in which there are more equations (corresponding to a large number of readings) than unknown numbers. The method of uniting the equations is called the method of *Least Squares.*

An observation equation, as here used, is an equation resulting from the substitution of readings in an equation assumed, or known, to be of the form connecting the measured variables. In such an equation coefficients are the unknowns. Thus, from Table VI, if x represents the hydraulic gradient and if y represents velocity, and if it is assumed that the law connecting y and x is linear, *i.e.*, $y = ax + b$, there result, by substituting the measured values of x and y, seven equations, viz.:

$$\begin{aligned}
16.9 &= 31.9\,a + b\\
11.4 &= 20.8\,a + b\\
24.5 &= 54.1\,a + b\\
4.3 &= 10.0\,a + b\\
10.1 &= 21.0\,a + b\\
58.8 &= 119.1\,a + b\\
22.7 &= 55.8\,a + b
\end{aligned}$$

a and b are two unknown numbers to be found, having given seven equations. The problem is not to find a set of values satisfying any two equations, but a set of values which will represent the most probable line upon which the points would fall if there were no errors of observation.

118. The Probable Error of a Single Reading. The probable error of a single reading, $E_{s.r.}$, is a number such that if it be subtracted from and then added to the arithmetic mean, the probability of a reading falling within these limits is exactly equal to 1/2, *i.e.*, the chance of a reading falling within these limits is exactly equal to the chance of its falling without. Graphically, the probable error determines two lines, as AB and $A'B'$, Fig. 99, parallel to and equidistant from the line of symmetry of the probability curve, such that the area between them, the curve, and the X-axis is one-half unit. (The area under the entire curve and the X-axis is unity.)

It is found that[1]

$$E_{s.r.} = 0.6745 \sqrt{\frac{\Sigma f v^2}{n}} = 0.6745\,\sigma$$

For the data given in Table XIV,

$$\sigma = 4.56.$$
$$E_{s.r.} = 3.08$$

which corresponds to 0.308 square centimeters.

119. The Probable Error of the Arithmetic Mean. If several sets of readings of the same magnitude were taken, the arithmetic means of these sets of readings would, in general, disagree among themselves. Thus, if one hundred students were each to take an equal number of readings, say one hundred, of the area bounded by a curve, the one hundred means of these sets would probably not all agree. From these one hundred means a frequency distribution polygon could be constucted; a mean (*i.e.*, a mean of the means) and a probable error could be calculated. This probable error, *i.e.*, the probable error of a single mean of the one hundred means, is called the probable error of the mean of any one of the sets of readings. The probable error, E, of the mean of a set of

[1] The value of

$$\frac{h}{\sqrt{\pi}} \int_0^x e^{-h^2x^2} dx$$

is found by expanding $e^{-h^2x^2}$ into an infinite series and integrating term by term Tables for different values of the upper limit, or rather for x/σ, or hx, are usually given in treatises on the Theory of Probability and in treatises on Statistical Methods. From such a table the value of 0.6745 for x/σ is found to correspond to 1/4.

readings, bears the same relation to a set of means as the probable error of a single reading bears to the readings of a particular set. A formula for the probable error of the mean is

$$E = \frac{E_{s.r.}}{\sqrt{n}} = \frac{0.6745}{n}\sqrt{\Sigma f v^2}.$$

The probable error of the mean of the readings of the area (Table XIV) is

$$E = \frac{3.08}{\sqrt{1068}} = 0.1,$$

or expressed in square centimeters $E = 0.01$. The probable error of a mean is usually indicated by affixing it with a double sign to the value of the mean. Thus, in the above expression,

$$\text{Area} = 127.66 \pm 0.01$$

This does *not* mean that the true area is somewhere between 127.65 and 127.67, but that with the data at hand the probability of the mean of a second set of the same number of readings falling within these limits is one-half (*i.e.*, the chances are one to one that it falls within these limits).

Exercises

1. Verify all numerical work done in connection with Table XIV.[1]

2. Using columns I and II, Table XIV, calculate h, σ, $E_{s.r.}$ and E. Arrange your work on a sheet of paper form *M*8 or *M*7

120. Biological Measurements. Frequently distributions of biological measurements are found to follow to a marked degree the probability law. While the distribution polygons may be asymmetric, or skew (See § **114**), the formulas for standard deviation, probable error, etc., are still applied as a method of comparison.[2]

Exercises

1. Construct the frequency polygon, find the mean, the standard deviation, the probable error of the mean, for the readings given in Table XV.

[1] In treatises on the Theory of Probability and in treatises on Statistical Methods are usually given tables of squares.

Crelle's or Peter's Multiplication Tables or the smaller table by Zimmermann may also be found very helpful in numerical calculations. A collection of tables and formulas is given in Statistical Methods by C. B. Davenport.

[2] At this point it is suggested that the student read pages 419–441, Principles of Breeding, by E. Davenport.

TABLE XV.—MEASUREMENTS OF THE LENGTH OF 327 EARS OF CORN[1]

Length of ear, inches	Number of ears, or frequency	Length of ear, inches	Number of ears, or frequency	Length of ear, inches	Number of ears, or frequency
v	f	v	f	v	f
3.0	1	6.5	8	9.5	63
3.5	0	7.0	12	10.0	38
4.0	1	7.5	19	10.5	21
4.5	0	8.0	32	11.0	8
5.0	2	8.5	40	11.5	2
5.5	3	9.0	67	12.0	1
6.0	9				

[1] These data are taken from Principles of Breeding, page 421, by E. Davenport.

CHAPTER X

[SMALL ERRORS]

121. Errors. Whenever an attempt is made to measure the magnitude of a quantity with an observing instrument, the readings thus obtained do not represent the true value of the given magnitude. There is a difference between the reading of the instrument, the observation, and the true value of the magnitude. This difference is called the error of the observation. Errors of observation may be due to many causes, some of which are: carelessness on the part of the observer or his inability to handle the measuring instrument properly; inadjustment of the measuring instrument; limitations due to the least count of the instrument, etc. By the **least count** of the measuring instrument is meant the smallest part of a unit which can be recorded with the instrument. To illustrate: when a volume of liquid is measured with a burette graduated to cubic centimeters and tenths of cubic centimeters, the readings are estimated to the nearest tenth of one-tenth of a cubic centimeter; the least count of the instrument is the one one-hundredth part of one cubic centimeter.

When direct measurements, which are in error, are used for calculating other magnitudes, these calculated magnitudes will in turn be in error. The larger the errors of the readings, the larger the errors of the calculated quantities; the smaller the errors of the readings, the smaller the errors of the calculated quantities. When the errors of the observations are known or admitted, it is at times desirable to know the corresponding error in the calculated magnitude obtained by substituting the readings in a formula. If, on the other hand, the calculated magnitude is desired only to a certain degree of accuracy, it is usually desirable to know the degree of accuracy to which the measurements must be made.

It will be apparent that the error of the calculated magnitude will depend upon the formula in which the readings are substituted.

Following are given the formulas for errors due to the simple algebraic operations and their applications to some of the formulas previously derived in this book.

122. Error of Product. The derivation of the formula for the error of the product of two numbers has been given in § **72,** but will be repeated here.

Let a and b be two readings. Let ϵ_1 and ϵ_2 be, respectively, their errors. The true values of the measured magnitudes are thus assumed to be $a \pm \epsilon_1$ and $b \pm \epsilon_2$. The product of the two magnitudes is then

$$(a \pm \epsilon_1)(b \pm \epsilon_2) \equiv ab \pm a\epsilon_2 \pm b\epsilon_1 + \epsilon_1\epsilon_2.$$

Since ab is the product obtained by multiplying the two readings together, the sum of the last three terms of the right-hand side of the above equation is the error of the product. Since ϵ_1 is small compared with a and ϵ_2 is small compared with b, $\epsilon_1\epsilon_2$ is very small compared with $a\epsilon_2 + b\epsilon_1$ and is negligible. Thus, if ϵ_1 and ϵ_2 are, respectively, the errors of a and b, $a\epsilon_2 + b\epsilon_1$ is the possible error of the product ab. The true product may fall anywhere between $ab + (a\epsilon_2 + b\epsilon_1)$ and $ab - (a\epsilon_2 + b\epsilon_1)$. All signs in the error are taken alike in order to obtain the greatest possible error.

In what follows, expressions of the form $a = 212 \pm 0.3$ will mean a reading of a equal to the first number (212), and a possible error equal to the second number (0.3).

If $a = 212 \pm 0.3$ and $b = 116 \pm 0.5$, the error in the product of the readings is $a\epsilon_2 + b\epsilon_1 = (212)(0.5) + (116)(0.3) = 150.8$. Thus, the true value of ab is between $24{,}592 + 151$ and $24{,}592 - 151$. In calculating the product ab by multiplying the two readings together there is no need of retaining more than the first three digits, and the product should be written 24,600.

Exercises

1. Show that, if ϵ_1, ϵ_2, and ϵ_3 are, respectively, the errors of a, b, and c, the error of abc is $ab\epsilon_3 + bc\epsilon_1 + ac\epsilon_2$.

Calculate the possible error of the product for each of the following:

2. $a = 217 \pm 0.2$; $b = 117 \pm 0.3$.

3. $a = 1267 \pm 0.5$; $b = 986 \pm 0.4$.

4. $a = 3176 \pm 0.3$; $b = 1012 \pm 0.3$.

5. $a = 3176 \pm 0.2$; $b = 1276 \pm 0.3$; $c = 987 \pm 0.4$.

123. Error of Quotient. Let a and b be the readings with errors of ϵ_1 and ϵ_2, respectively. The quotient of the true magnitudes is

$$\frac{a \pm \epsilon_1}{b \pm \epsilon_2} = \frac{a}{b} + \frac{\pm \frac{a}{b}\epsilon_2 \pm \epsilon_1}{b \pm \epsilon_2} = \frac{a}{b} + \frac{\pm a\epsilon_2 \pm b\epsilon_1}{b(b \pm \epsilon_2)}.$$

Since $\frac{a}{b}$ is the quotient of the two readings, the second term of the right-hand side of the above equation is the error of the quotient. Since ϵ_2 is small compared with b, the quantity $b + \epsilon_2$ may be replaced by b, and the possible error of the quotient becomes

$$\frac{a\epsilon_2 + b\epsilon_1}{b^2}.$$

If $a = 317 \pm 0.5$ and $b = 110 \pm 0.6$, the error of the quotient $\frac{a}{b}$ is

$$\frac{(317)(0.6) + (110)(0.5)}{(110)^2} = 0.02.$$

Exercises

Find the possible error of the quotient for each of the following:

1. $a = 1127 \pm 0.5$; $b = 56 \pm 0.2$.
2. $a = 2216 \pm 0.6$; $b = 27 \pm 0.1$.
3. $a = 2716 \pm 0.3$; $b = 25 \pm 0.5$.

124. Error of the Square. Let ϵ be the error of the reading a. The square of the true value is $(a \pm \epsilon)^2 = a^2 \pm 2a\epsilon + \epsilon^2$. Since a^2 is the square of the reading, the sum of the last two terms of the right-hand member of the above equation is the error of the square. The possible error, $2a\epsilon + \epsilon^2$, may be written $(2a + \epsilon)\epsilon$, and, since ϵ is small compared with $2a$, the expression within the parentheses reduces to $2a$ without appreciable error. The possible error of the square is then

$$2a\epsilon.$$

If $a = 112 \pm 0.5$, the error of a^2 is $(2)(112)(0.5) = 112$. Since a^2 equals 12,544, the error of 0.5 in a, or about 1/2 of 1 percent, produces an error of about 1 percent in a^2.

Exercises

Calculate the error and percent error of the square for each of the following:

1. $a = 111 \pm 0.1$.

2. $a = 316 \pm 0.2$.

3. $a = 416 \pm 0.6$.

4. Show that the percent error of the square of a reading is the double of the percent error of the reading.

5. Show that the error of the cube of $a \pm \epsilon$ is $3a^2\epsilon$.

6. Show that the percent error of the cube of a reading is the treble of the percent error of the reading.

7. Show that the error of the fourth power of $a \pm \epsilon$ is $4a^3\epsilon$.

125. Error in the Square Root. Let ϵ be the error of the reading a. The square root of the true value is then

$$(a \pm \epsilon)^{\frac{1}{2}} \equiv a^{\frac{1}{2}} \pm \frac{1}{2}a^{-\frac{1}{2}}\epsilon + \frac{1}{8}a^{-\frac{3}{2}}\epsilon^2 \pm \frac{1}{16}a^{-\frac{5}{2}}\epsilon^3 + \cdots$$

by the binomial theorem. (See § **100.**) The above may be written

$$(a \pm \epsilon)^{\frac{1}{2}} = a^{\frac{1}{2}} \pm \frac{\epsilon}{2\sqrt{a}}\left[1 \pm \frac{\epsilon}{4a} + \frac{\epsilon^2}{8a^2} \pm \cdots\right].$$

Since ϵ is small compared with a, the terms within the parentheses, excepting the first, are small compared with 1, and may be neglected. Hence

$$(a \pm \epsilon)^{\frac{1}{2}} = a^{\frac{1}{2}} \pm \frac{\epsilon}{2\sqrt{a}}$$

approximately. Since $a^{\frac{1}{2}}$ is the square root of the reading, $\frac{\epsilon}{2\sqrt{a}}$ is the error of the square root of a.

If $a = 121 \pm 0.5$, the error of $\sqrt{a}$ is

$$\frac{0.5}{(2)(11)} = 0.023.$$

Exercises

Find the error of the square root of each of the following:

1. $a = 121 \pm 0.1$.

2. $a = 146 \pm 0.3$.

3. $a = 216 \pm 0.6$.

4. Show that the percent error of the square root of a reading is one-half of the percent error of the reading.

5. Show that the error of the cube root of a reading is $\frac{\epsilon}{3a^{\frac{2}{3}}}$.

6. Show that the percent error of the cube root of a reading is one-third the percent error of the reading.

7. Show that the error of the fourth root of a reading a is $\frac{\epsilon}{4a^{\frac{3}{4}}}$.

126. Error of Sine α. Let ϵ, expressed in radians, be the error of the measured angle α. The sine of the true angle is then $\sin(\alpha \pm \epsilon)$. Let BOA, Fig. 102, be the angle α with its vertex at the center, O, of a circle having a unit radius, OA. BE is the sine of the reading α, CD is the sine of the true angle, and CF is the error of the sine due to the error ϵ of the reading α. The angle FCB is very nearly equal to the angle α, and approaches it as ϵ approaches zero. The chord CB is very nearly equal to the arc CB. FC is then very nearly equal to the product of the chord CB by the cosine of FCB, or the product of the arc CB by the cosine of α. But, since ϵ is expressed in radians, the arc CB equals the product of ϵ by the radius of the circle, which is unity. We thus have $\mathbf{CF = \epsilon \cos \alpha}$, approximately. The figure is drawn for ϵ positive; the same conclusion is obtained if ϵ be taken negative. The sign of ϵ regulates the sign of the error of the sine of α.

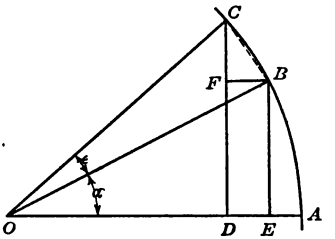


FIG. 102.

To illustrate the above formula, suppose $\alpha = 30° \pm 15'$. The error $15'$ expressed in radians is

$$\frac{15}{60}\frac{\pi}{180} = 0.00436.$$

The error then of the sine α is $(0.00436)\cos 30°$,

or $$(0.00436)(0.866) = 0.0038.$$

This error (or the error of any trigonometric function of any angle) may also be obtained from Table XXII. the table of

natural trigonometric functions. The sine of 30° is 0.5, the sine of 30° + 15′ is 0.5038, and the sine of 30° − 15′ is 0.4962. In each case the error in magnitude is 0.0038, the same as found above by using the formula.

Exercises

1. Find by substituting in the formula the error of sin α for each of the following:

(*a*) $\alpha = 27° \pm 10'$.

(*b*) $\alpha = 87° \pm 20'$.

2. Find by means of the table of natural trigonometric functions the error of sine α for each angle given in exercise 1.

3. Find the error of cos α, if $\alpha = 26° \pm 25'$.

4. Find the error of tan α, if $\alpha = 62° \pm 17'$.

127. Error of the Area of a Triangle Computed from Two Sides and the Included Angle. The formula for the area, A, of a triangle in terms of two sides, a and b, and the included angle, γ, is

$$A = \tfrac{1}{2}\, ab \sin \gamma.$$

Let ϵ_1 ϵ_2, ϵ_3, be the errors of a, b, and γ, respectively. Let E represent the error of sin γ. Then, by exercise 1, § **122,** the error of A is $\frac{1}{2}\,(abE + b\epsilon_1 \sin \gamma + a\epsilon_2 \sin \gamma)$,[1] where $E = \epsilon_3 \cos \gamma$. In this formula, ϵ_3 must be expressed in radians. To illustrate: let $a = 111 \pm 0.2$ rods; $b = 72 \pm 0.2$ rods, and $\gamma = 37° \pm 15'$. From the table of natural functions, E is found to be 0.0035. The error of A is then

$\frac{1}{2}\left[(111)\ (72)\ (0.0035) + (72)\ (0.6018)\ (0.2) + (111)\ (0.6018)\ (0.2)\right]$, or 50 square rods.

Exercises

Find the error of the area of the triangle for each of the following:

1. $a = 126 \pm 0.3$ rods; $b = 137 \pm 0.3$ rods; $\gamma = 47° \pm 10'$.

2. $a = 363 \pm 0.8$ rods; $b = 216 \pm 0.5$ rods; $\gamma = 26° \pm 15'$.

[1] The value of E may be taken from a table of natural trigonometric functions, or the expression given for it in the preceding section may be substituted, in which case the expression for the error of A becomes

$$\tfrac{1}{2}(ab\epsilon_3 \cos \gamma + b\epsilon_1 \sin \gamma + a\epsilon_2 \sin \gamma).$$

128. Error of the Area of a Triangle Computed from Three Sides. The formula for the area, A, of a triangle in terms of three sides is

$$A = \sqrt{s(s-a)(s-b)(s-c)},$$

where $s = \frac{1}{2}(a+b+c)$. This formula may be put in the form

$$A = \tfrac{1}{4}\sqrt{2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4}$$

Let ϵ_1, ϵ_2, and ϵ_3 be the errors of a, b, and c, respectively. Then, by § **124**, the error of a^2 is $2a\epsilon_1$, of b^2 is $2b\epsilon_2$, of c^2 is $2c\epsilon_3$, of a^4 is $4a^3\epsilon_1$, of b^4 is $4b^3\epsilon_2$ and of c^4 is $4c^3\epsilon_3$. By § **122**, the error of a^2b^2 is $2a^2b\epsilon_2 + 2ab^2\epsilon_1$, of b^2c^2 is $2bc^2\epsilon_2 + 2b^2\epsilon_3$, and of c^2a^2 is $2ca^2\epsilon_3 + 2c^2a\epsilon_1$. The error of the quantity under the radical sign is then

$$4a^2b\epsilon_2 + 4ab^2\epsilon_1 + 4b^2c\epsilon_3 + 4bc^2\epsilon_2 + 4c^2a\epsilon_1 + 4ca^2\epsilon_3 - 4a^3\epsilon_1 - 4b^3\epsilon_2 - 4c^3\epsilon_3,$$

or

$$4\left[a(b^2+c^2-a^2)\epsilon_1 + b(c^2+a^2-b^2)\epsilon_2 + c(a^2+b^2-c^2)\epsilon_3\right].$$

By § **125** the error of A is found to be

$$\frac{1}{4} \times \frac{4\left[a(b^2+c^2-a^2)\epsilon_1 + b(c^2+a^2-b^2)\epsilon_2 + c(a^2+b^2-c^2)\epsilon_3\right]}{2\sqrt{2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4}},$$

or

$$\frac{\mathbf{a}(\mathbf{b}^2+\mathbf{c}^2-\mathbf{a}^2)\epsilon_1 + \mathbf{b}(\mathbf{c}^2+\mathbf{a}^2-\mathbf{b}^2)\epsilon_2 + \mathbf{c}(\mathbf{a}^2+\mathbf{b}^2-\mathbf{c}^2)\epsilon_3}{\mathbf{8A}}.$$

If ϵ_1, ϵ_2, and ϵ_3 are each $\mathbf{k}$ percent of the length of the sides of which they are errors, we have

$$\epsilon_1 = \frac{\mathbf{ka}}{\mathbf{100}}$$

$$\epsilon_2 = \frac{\mathbf{kb}}{\mathbf{100}}$$

$$\epsilon_3 = \frac{\mathbf{kc}}{\mathbf{100}}$$

Substituting these values in the above expression for the error of A, it reduces to

$$\frac{k}{800A}\left[a^2(b^2+c^2-a^2) + b^2(c^2+a^2-b^2) + c^2(a^2+b^2-c^2)\right],$$

or

$$\frac{k}{800A}(2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4),$$

or

$$\frac{k}{800A}16A^2, \text{ or } \frac{2kA}{100}.$$

This expression says that if the same percent of error, k, is made in measuring the three sides of a triangle, the error of the calculated area is $2k$ percent. It is to be noted that in order for this statement to be true the three errors must have the same algebraic sign.

To illustrate: if $a = 100 \pm 0.1$ foot, $b = 200 \pm 0.2$ foot, and $c = 150 \pm 0.1$ foot, the posible error of A is

$$\frac{525{,}000 + 300{,}000 + 412{,}500}{84},$$

or 21 square feet or a possible error of about 3/10 of 1 percent of the area.

Exercises

1. Find the possible error of the area of the triangle if $a = 127 \pm 0.2$ rod, and $b = 263 \pm 0.4$ rod, and $c = 290 \pm 0.5$ rod.

2. In a triangle $a = 100 \pm 0.3$, $\beta = 37° \pm 15'$, and $\gamma = 52° \pm 15'$. Calculate the possible error of b.

HINT: Use the law of sines and note that the possible error of α is the sum of the errors of β and γ.

3. If the error of α is ϵ, expressed in radians, show that the error of $\cos \alpha$ is $\epsilon \sin \alpha$.

4. If $a = 100 \pm 0.2$, $b = 200 \pm 0.3$, and $\alpha = 34° \pm 15'$, find the possible error of

$$\sqrt{\frac{a^2 + b^2 \sin^2 \alpha}{\cos \alpha}}.$$

CHAPTER XI

[POINT, PLANE, AND LINE IN SPACE]

129. Rectangular Space Coördinates. Through any point O, Fig. 103, are drawn three right lines, $X'X$, $Y'Y$, and $Z'Z$, mutually at right angles. The point O is called the **origin** of coördinates. The three lines $X'X$, $Y'Y$, and $Z'Z$ are called, respectively, the X-axis, the Y-axis, and the Z-axis of coördinates. The three planes determined by the three axes taking them two by two are called the **coördinate planes.** The coördinate planes are mutually at right angles. The plane determined by the X- and Y-axes is called the XY-plane, by the Y- and Z-axes the YZ-plane, and by the X- and Z-axes the XZ-plane.

The three coördinate planes divide space into eight parts called **octants.** The octants are numbered from one to eight. The first octant is in front of the XY-plane, to the right of the YZ-plane, and above the XZ-plane. The second octant is back of the XY-plane, to the right of the YZ-plane, and above the XZ-plane. The third octant is back of the XY-plane, to the left of the YZ-plane, and above the XZ-plane. The fourth octant is in front of the XY-plane, to the left of the YZ-plane, and above the XZ-plane. The fifth, sixth, seventh, and eighth octants are below the first, second, third and fourth octants, respectively.

Distances measured along, or parallel to, the X-axis to the right from O are called positive, to the left from O negative. Distances measured along, or parallel to, the Y-axis up from O are called positive, down from O negative. Distances measured along, or parallel to, the Z-axis out from O are called positive, back from O negative.

Any point, P, in space is located by giving its distance from each of the three coördinate planes. FP, or OA, Fig. 103, its distance from the YZ-plane, is called its x-coördinate; DP, or OC, its distance from the XZ-plane, is called its y-coördinate;

and EP, or OB, its distance from the XY-plane, is called its z-coördinate. The algebraic signs of the three coördinates of points in the eight octants are given in the following table:

Octant	x	y	z
First	+	+	+
Second	+	+	−
Third	−	+	−
Fourth	−	+	+
Fifth	+	−	+
Sixth	+	−	−
Seventh	−	−	−
Eighth	−	−	+

Since the axes of coördinates are taken mutually at right angles, the three coördinates of a point P are called rectangular coördinates. In designating a point its three rectangular coördinates are written in the order: the x-coördinate first, then the y-coördinate, and then the z-coördinate. Thus, the point (1, 2, 3) is in the first octant, one unit from the YZ-plane, two units from the XZ-plane, and three units from the XY-plane. The point $(-1, -2, 3)$ is in the eighth octant, one unit from the YZ-plane, two units from the XZ-plane, and three units from the XY-plane. The point $(-1, -1, -2)$ is in the seventh octant.

Exercises

Locate the following points:

1. (1, 1, 1). **2.** (3, − 2, 1). **3.** (− 2, − 1, − 5).
4. (3, − 2, − 3). **5.** (− 3, 4, 3).

130. Distance between Two Points. The distance of P, Fig. 103, from the origin is the length of the diagonal of the rectangular parallelopiped whose edges are x, y, z, the coördinates of P. The length of this diagonal, or the distance of P from the origin is $\sqrt{x^2 + y^2 + z^2}$.

Let P_1 and P_2, Fig. 104, be any two points whose coördinates

are, respectively, x_1, y_1, z_1, and x_2, y_2, z_2. The student will show that the distance between P_1 and P_2 is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

131. The Locus of an Equation.[1] In geometry of two dimensions the equation $x = 2$ represents a straight line parallel to the Y-axis, two units to its right. (See § **26.**) In geometry of three dimensions the same equation, $x = 2$, represents a plane

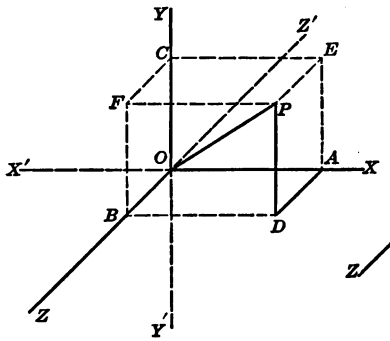


FIG. 103.

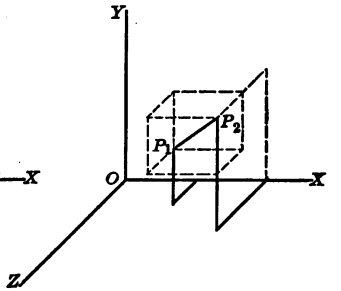


FIG. 104.

parallel to the YZ-plane, two units to its right. For, the only condition to be satisfied is that the y-coördinate of the point is $+ 2$; the coördinates of all points of the plane satisfy this condition, and the coördinates of a point not of the plane do not satisfy the condition. The locus of all points satisfying the condition $x = 2$ is a plane parallel to the YZ-plane two units to its right.

The locus of a point satisfying the condition $x = 5$ is a plane parallel to the YZ-plane five units to its right. $x = -2$ represents a plane parallel to the YZ-plane two units to its left. $x = 0$ represents the YZ-plane. $x = k$, where k is any constant, represents a plane parallel to the YZ-plane k units from it; to the right if k is positive, to the left if k is negative. Similarly, $y = k$ represents a plane parallel to the XZ-plane, and $z = k$ represents a plane parallel to the XY-plane.

[1] The locus of an equation is the locus of a point whose coördinates, x, y, and z, satisfy the equation.

Any equation between x, y, and z, when solved for z, takes the form $z = f(x, y)$. (The right-hand member of this equation, $f(x, y)$, is read "function of x and y," and is an abbreviation for the right-hand side of the equation when solved for z. The $f(x, y)$ may contain x's and y's, but it does not contain z.) If, when to x and y in the equation $z = f(x, y)$ any two values are assigned, there corresponds a real value of z, the three values, one for x, one for y, and one for z, locate a point in space whose coördinates satisfy the equation. If a second set of values is assigned to x and y, and a second, or the same, value for z found, a second point is located in space. If the difference between the two x-values and the difference between the two y-values is made less and less, the difference between the two z-values will in general grow less and less. The locus of a point whose coördinates satisfy an equation is then seen to be a surface. Thus, in the equation

$$z = \pm\sqrt{81 - (x^2 + y^2)}$$

if any set of values is assigned to x and y such that $x^2 + y^2 < 81$, there correspond two real values of z, numerically equal but with opposite signs. If a second set of values is assigned to x and y, a second (or the same) pair of values for z is obtained. If the difference between the two x-values and the difference between the two y-values are made smaller and smaller, the difference between the two positive values and between the two negative values of z are made smaller and smaller. As x and y are made to vary continuously, z varies continuously and the equation represents a surface in space. If the values assigned to x and y are such that $x^2 + y^2 > 81$, the corresponding values of z are imaginary, which means that the surface does not extend to a point whose x and y are the values assigned to x and y in the equation. Thus, if x and y are each given the value 7 the equation reduces to $z = \pm\sqrt{-17}$, an imaginary. This says that the surface represented by the equation contains no point whose x- and y-coördinates are each equal to 7, or in other words, a line parallel to the z-axis seven units each from the XZ- and YZ-coördinate planes does not pierce nor touch the surface.

The equation $z = \pm\sqrt{81 - (x^2 + y^2)}$ may be put in the form $x^2 + y^2 + z^2 = 81$. The expression $x^2 + y^2 + z^2$ is the square of

the distance from the origin to the point whose coördinates are x, y and z. (See § 130.) This expression being a constant, 81, shows that all points whose coördinates satisfy the equation must be at a distance 9 from the origin. The locus of the equation is then a sphere with center at the origin and radius equal to 9.

132. The Equation of a Locus. The equation of the locus of a point is an equation satisfied by the coördinates of all points of the locus and by the coördinates of no other point. Thus, the equation of a sphere, center at the origin and radius equal to 10, is $x^2 + y^2 + z^2 = 100$. The equation of the XZ-plane is $y = 0$. The equation of the XY-plane is $z = 0$. The equation of the YZ-plane is $x = 0$. The equation of the XY- and XZ-planes considered as a single locus is $yz = 0$. The equation of the three coördinate planes considered as a single locus is $xyz = 0$.

Exercises

1. Find the distance between the following pairs of points:

(*a*) (1, 1, 1) and (3, 2, 1). (*b*) (3, − 2, 6) and (− 2, 1, 7).
(*c*) (− 1, − 2, − 3) and (2, 6, − 1).

2. What loci are represented by the following equations:

(*a*) $x = 3$. (*b*) $x = 0$. (*c*) $y = -6$.
(*d*) $y = 0$. (*e*) $z = 0$. (*f*) $z = -10$.
(*g*) $x^2 + y^2 + z^2 = 1$. (*h*) $x^2 + y^2 + z^2 = 25$.
(*i*) $xy = 0$. (*j*) $xyz = 0$. (*k*) $x = y$.
(*l*) $x = z$. (*m*) $y = z$.

3. Find the equations of the locus of (*a*) a point ten units from the origin; (*b*) a point ten units from the point (3, 1, 2); (*c*) a point equidistant from the XZ- and the YZ-coördinate planes.

133. The Direction Cosines of a Line. Let P, Fig. 103, be any point on the line OP passing through the origin. The projections of OP upon the three coördinate axes are the coordinates of the point P. Let α, β, and γ be the angles, called **the direction angles of the line OP,** between OP and the positive directions of the x-, y-, and z-axes, respectively. The cosines of these angles are called the **direction cosines** of the line OP.

The direction angles of a line through the origin are positive and do not exceed 180°. From the figure it is seen that

$$x = OP \cos \alpha$$
$$y = OP \cos \beta$$
$$z = OP \cos \gamma$$

Squaring and adding these three equations,

$$x^2 + y^2 + z^2 = OP^2(\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma).$$

But, since

$$OP^2 = x^2 + y^2 + z^2,$$

it follows that

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1. \qquad (1)$$

The direction angles of a line not passing through the origin are defined as the direction angles of a parallel line passing through the origin.

The sum of the squares of the direction cosines of a line is equal to unity.

134. The Projection of a Broken Line upon a Given Line. Let $ABCDE$, Fig. 105, be any broken line in the plane or in

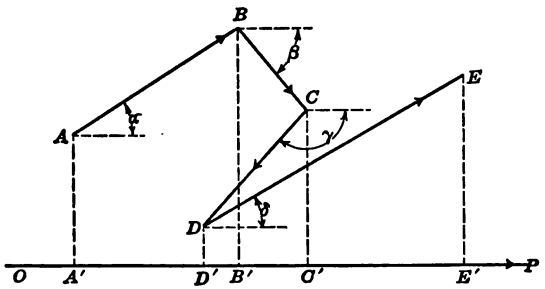


FIG. 105.

space. From the points A, B, C, D, and E perpendiculars are dropped upon the line OP meeting it at the points A', B', C', D', and E'. $A'B'$, $B'C'$, $C'D'$, and $D'E'$ are, respectively, the projections of AB, BC, CD, and DE. Let α, β, γ, and δ be the angles between the directed segments of AE and the line OP. $A'B' = AB \cos \alpha$, $B'C' = BC \cos \beta$, $C'D' = CD \cos \gamma$ (which is negative, as γ is obtuse) and $D'E' = DE \cos \delta$. Then $A'E'$,

the projection of a directed broken line upon any directed line, is the sum of the products of the length of each segment of the broken line by the cosine of the angle between it and the directed line upon which it is projected.

135. The Normal Equation of a Plane. A plane is determined if its normal distance, p, from the origin, and the direction angles of this normal are known. Let ON, or p, Fig. 106, be the normal

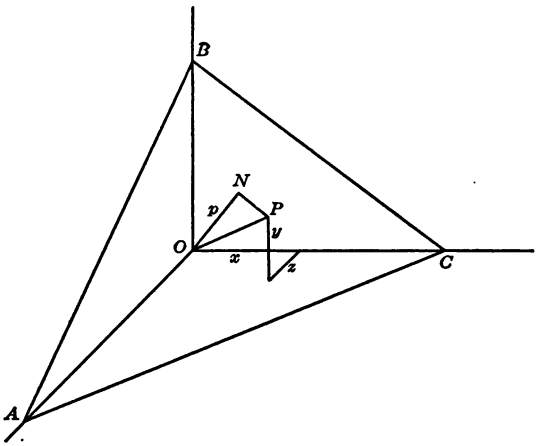


FIG. 106.

drawn from the origin to the plane ABC. Let α, β, and γ be the direction angles of this normal. Let P be any point in the plane. Draw PN, which is perpendicular to ON. The sum of the projections of x, y, z, and PN upon the normal is equal to p. The projection of x upon the normal is $x \cos \alpha$, of y is $y.\cos \beta$, and of z is $z \cos \gamma$. The projection of PN upon the normal is zero, since ONP is a right angle. It then follows that

$$x \cos \alpha + y \cos \beta + z \cos \gamma = p, \tag{2}$$

which is called the **normal equation of the plane.**

In the equation $Ax + By + Cz = D$, let A, B, C, and D be any real constants. The equation is then the general equation of the

first degree in x, y, and z. Upon dividing by $\pm\sqrt{A^2+B^2+C^2}$, this equation reduces to

$$\frac{A}{\pm\sqrt{A^2+B^2+C^2}}x+\frac{B}{\pm\sqrt{A^2+B^2+C^2}}y+\frac{C}{\pm\sqrt{A^2+B^2+C^2}}z=\frac{D}{\pm\sqrt{A^2+B^2+C^2}}. \quad (3)$$

The sign before the radical is to be chosen so as to make the right-hand side of the equation positive.

Since the numerical values of the coefficients of x, y, and z in equation (3) cannot exceed unity, they are each the cosine of some angle. Call these angles, respectively, α, β, and γ. Then

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=\frac{A^2}{A^2+B^2+C^2}+\frac{B^2}{A^2+B^2+C^2}+\frac{C^2}{A^2+B^2+C^2}=1,$$

α, β and γ are then, by § **133**, the direction cosines of some line. The left-hand side of (3) reduces to

$$x\cos\alpha+y\cos\beta+z\cos\gamma,$$

the left-hand side of the normal equation of a plane; and since the right-hand side of (3) is a positive quantity, call it p, equation (3) reduces to

$$\mathbf{x\cos\alpha+y\cos\beta+z\cos\gamma=p},$$

the normal equation of a plane.

Every equation of a plane is of the first degree in x, y, and z; and, conversely, every equation of the first degree in x, y, and z is the equation of a plane. To reduce $Ax+By+Cz=D$ to the normal form, divide the equation through by $\pm\sqrt{A^2+B^2+C^2}$, choosing that sign before the radical which will make the right-hand side positive.

To illustrate: let $x+2y-3z=-2$ be the given equation. To reduce it to the normal form divide by $-\sqrt{14}$, which gives

$$-\frac{1}{\sqrt{14}}x-\frac{2}{\sqrt{14}}y+\frac{3}{\sqrt{14}}z=\frac{2}{\sqrt{14}}.$$

The plane is $\frac{2}{\sqrt{14}}$ units from the origin. The $\cos\alpha$, $-\frac{1}{\sqrt{14}}$, is negative, which says that the normal drawn from the origin to the plane extends into the third, fourth, seventh or eighth octant. The $\cos\beta$, $-\frac{2}{\sqrt{14}}$, is negative, which says that the normal from the origin to the plane extends into the fifth, sixth, seventh or eighth octant. The $\cos\gamma$, $+\frac{3}{\sqrt{14}}$, is positive, which says that the normal drawn from the origin to the plane extends into the first, fourth, fifth or eighth octant. The three direction cosines, then, show that the normal drawn from the origin to the plane extends into the eighth octant; or the plane and the three coördinate planes form a tetrahedron in the eighth octant.

Exercises

Reduce each of the following equations of planes to the normal form, find the distance of each plane from the origin, and locate the octant in which the plane forms a tetrahedron with the coördinate planes.

1. $x + y + z = 1$.
2. $x - y + z = 6$.
3. $3x - 2y + 5z - 6 = 0$.
4. $2x + 7y - z + 7 = 0$.

136. The Slope of a Plane, Given the Elevation of Three Points. In this section is described a graphic method for determining the slope, both in magnitude and direction, of a plane surface, if the relative elevations of any three of its points are known. A description and an illustration of the method will be given before the proof.

Let A, B, and C, Fig. 107, be the three points whose elevations are known. Suppose A the highest. Compute the gradient (in feet per 100 feet, in feet per 1000 feet, in feet per mile, or in any other desired unit) from A to B and from A to C. From A toward B draw AB', whose length, drawn to scale, represents the gradient from A to B. From A toward C draw AC', whose length, to scale, represents the gradient from A to C. At the points B' and C' draw lines, respectively, perpendicular to AB and AC.

These perpendiculars intersect at the point X. Draw AX, which represents in direction and magnitude the slope of the plane ABC.

To illustrate: suppose B and C are, respectively, 2.26 and 3.20 feet lower than A. Suppose $AB = 838$ feet, and $AC = 866$ feet (measured along the horizontal). The gradient of the plane ABC in the direction AB is 2.7 feet per 1000 feet; and in the direction AC, 3.7 feet per 1000 feet. Lay off AB' and AC', respectively equal

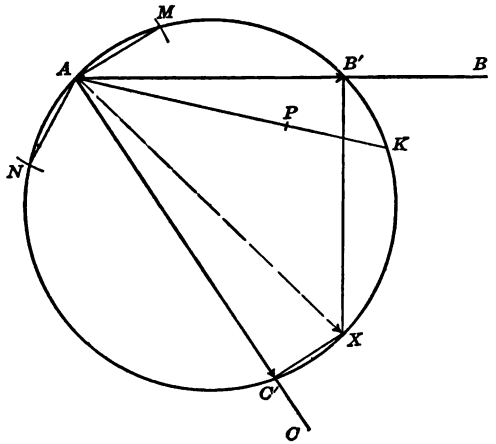


FIG. 107.

to 2.7 and 3.7 inches, thus making 1 inch represent a slope of 1 foot per 1000 feet.[1] AX is 3.79 inches long, which shows that any line on the plane parallel to AX has a gradient of 3.79 feet per 1000 feet. The direction AX is that of greatest slope.

If A, B and C were points upon the plane surface of an uneroded sedimentary rock, the length of AX would represent the dip, and the direction perpendicular to AX would be the strike of the stratum.

If A, B and C represent points upon the ground-water plane, AX would represent the direction in which the water would be seeping.

[1] The drawing here given is reduced. The student should redraw it to scale, and check the results given here.

To lay out the direction of a line having a given slope, say 1 foot per 1000 feet, proceed as follows: Upon AX as diameter describe a circle. With A as center and with a radius equal to 1 inch describe arcs cutting the circle at the points M and N. AM and AN are then the directions having a drop of 1 foot per 1000 feet.

To find the elevation of any point P, draw AP intersecting the circle $AB'C'X$ at the point K. The length AK gives the slope of the plane in the direction AP. With this slope and the distance AP the elevation of P relative to A may be calculated. In our specific illustration, AK is 3.25 inches, indicating a drop of 3.25 feet per 1000 feet in the direction AP. The length AP is 2.18 inches, making P 436 feet distant from A. (The scale of ABC is 1 inch to 200 feet.) From these two measurements P is found to be 1.42 feet lower than A.

Exercises

1. A, B and C are three points on a plane. $AB = 500$ feet, $AC = 600$ feet, $BC = 400$ feet, measured on the horizontal. The elevations of the plane at the points A, B, and C are, respectively, 137.6 feet, 132.4 feet, and 135.7 feet. Find graphically the magnitude and direction of the slope of the plane ABC. Find the elevations of two points each 300 feet from B and 200 feet from C. Find two directions having a slope of 3 feet per 1000 feet. Find the slope in the directions from A to two points each 300 feet from A and 400 feet from B.

[137.] Proof of the Construction Given in the Preceding Section. In the proof for the above construction let A, Fig. 108, be taken as the origin of coördinates, and let the XY-plane pass through the point B. The coördinates of A are then 0, 0, and 0. Let the coördinates of B be a, $-b$, and 0; and of C be a', $-b'$, and c'. The general equation of a plane, by § **135**, is

$$Ax + By + Cz = D. \tag{4}$$

Since the plane passes through the origin, (0, 0, 0) must satisfy its equation, which shows that D is zero. Since the plane passes through the point B, its coördinates must satisfy the equation, thus giving

$$aA - bB = 0. \tag{5}$$

Similarly, the coördinates of the point C must satisfy the equation, giving

$$a'A - b'B + c'C = 0. \tag{6}$$

Solving equations (5) and (6) for A and B we find

$$A = \frac{bc'C}{ab' - a'b}$$

and

$$B = \frac{ac'C}{ab' - a'b}.$$

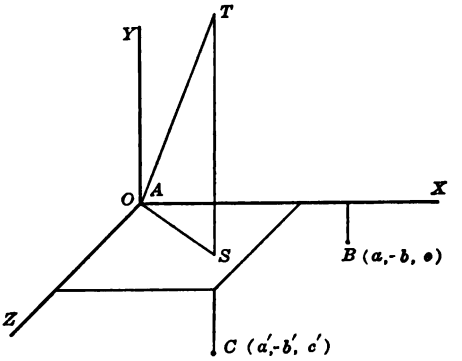


FIG. 108.

With these values, equation (4) reduces to

$$bc'x + ac'y + (ab' - a'b)z = 0. \tag{7}$$

The direction cosines of the upward drawn normal to this plane are, by § **133**,

$$\frac{bc'}{\sqrt{I}}, \quad \frac{ac'}{\sqrt{I}}, \text{ and } \frac{ab' - a'b}{\sqrt{I}}$$

where

$$I = b^2c'^2 + a^2c'^2 + (ab' - a'b)^2.$$

The gradient of the line AB is $\frac{b}{a}$, and of AC is $\frac{b'}{\sqrt{a'^2 + c'^2}}$. The equations of the lines $B'X$ and $C'X$, Fig. 107, on the XZ-plane are, respectively, $x = \frac{b}{a}$ and $a'x + c'z = b'$. The coördinates of the point X are then $x = \frac{b}{a}$ and $z = \frac{ab' - a'b}{ac'}$.

Let AT, Fig. 108, be the upward drawn normal to the plane. Take T such that $AT = \frac{\sqrt{I}}{AC'}$. The coördinates of S, the projection of T upon the XZ-plane, are $\frac{\sqrt{I}}{ac'}\ \frac{bc'}{\sqrt{I}}$ and $\frac{\sqrt{I}}{ac'}\ \frac{ab'-a'b}{\sqrt{I}}$, or, $\frac{b}{a}$ and $\frac{ab'-a'b}{ac'}$, which are seen to be the coördinates of the point X. This shows that the slope of the plane is in the direction AX. We shall now show that the length of AX represents the magnitude of the slope of the plane.

The distance AX is $\frac{1}{ac'}\sqrt{b^2c'^2+(ab'-a'b)^2}$. By substituting the coördinates of X in the equation of the plane passing through the points A, B, and C, we find the plane to be

$$\frac{1}{a^2c'^2}\left[b^2c'^2+(ab'-a'b)^2\right]$$

units directly beneath X. Upon dividing this expression by the length of AX we have

$$\frac{1}{ac'^2}\sqrt{b^2c'^2+(ab'-a'b)^2},$$

which is equal to the length of AX found above. This shows that AX represents the magnitude of the slope of the plane ABC.

Since any angle inscribed within a semicircle is a right angle, a perpendicular drawn to AK, Fig. 107, at the point K will pass through X. From what was proved above, AK must represent the gradient of the plane ABC in the direction AP. For the same reason, AM represents the slope (given) in the direction AM.

CHAPTER XII

[MAXIMA AND MINIMA]

138. Maxima and Minima Defined. Let y be a function of x (See § **45**). If, as x increases, y increases to a certain value, M, and then begins to decrease, M is called a **maximum** value of y. If, as x increases, y decreases to a certain value, m, and then begins to increase, m is called a **minimum** value of y.

A maximum value of a function is not necessarily the largest value the function may have; and a minimum value is not necessarily its smallest value. A function may have more than one maximum, and more than one minimum, and a function may have neither maxima nor minima.

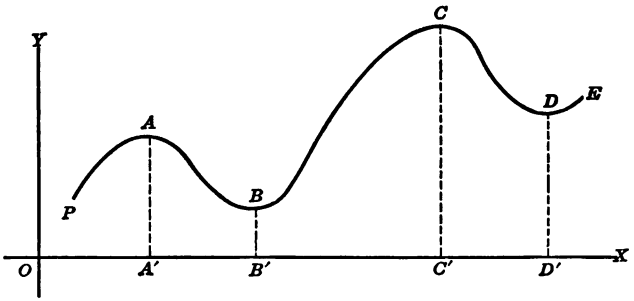


FIG. 109.

Graphically the maxima and minima values of a function are, respectively, the lengths of the ordinates of the high and low points of the graph of the function.

Let $ABCDE$, Fig. 109, be the graph of the function $y = f(x)$. As x increases from zero to OA', y increases to the value represented by the ordinate $A'A$. As x increases from OA' to OB', y decreases. Then by definition, $A'A$ represents a maximum value of the function. Similarly $C'C$ represents a second maxi-

mum value, and $B'B$ and $D'D$ represent two minimum values. The minimum value DD' is larger than the maximum value $A'A$.

139. Maximum and Minimum Values by Plotting. When observed data connecting two variables are plotted upon rectangular coördinate paper, and a smooth curve is drawn through the plotted points, the maximum and minimum values may be found approximately from the graph. Thus, from Fig. 15. the maximum air temperature was approximately 67.5° at 3 : 45,

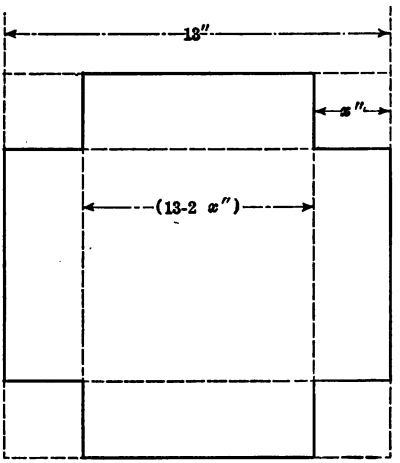


FIG. 110.

If the equation of the curve be known it may be plotted point by point and the approximate values of the maximum and minimum values taken from the graph. To illustrate this method take the following problem: From a square piece of tin 13 inches on a side an equal square is cut from each corner. The edges are then turned up forming a box. Find the size of the square cut from each corner such that the volume of the box shall be a maximum.

Let y be the volume of the box. Let x be the side of the square cut out from each corner, Fig. 110. The volume of the box is then $y = (13 - 2x)^2x$. By assigning to x values and calculating the corresponding values of y the following table is constructed:

x	y	x	y
0	0	7/2	126
1/2	72	8/2	100
2/2	121	9/2	72
3/2	150	10/2	45
4/2	162	11/2	22
5/2	160	12/2	6
6/2	147	13/2	0

From this table it is seen that a maximum value of y exists when x lies somewhere between $\frac{3}{2}$ and $\frac{5}{2}$. By giving to x, values nearer together, the following table is constructed:

x	y	x	y
1.6	153.7	2.6	158.2
1.8	159.0	2.8	153.3
2.0	162.0	3.0	147.0
2.2	162.7	3.2	139.4
2.4	161.4	3.4	130.7

From this table it is seen that the maximum value of y occurs for some value of x between 2.0 and 2.4 In Fig. 111 is given the

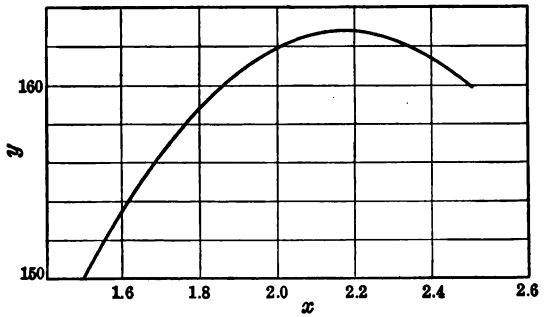


FIG. 111.

plot of the function for values of x ranging from $\frac{3}{2}$ to $\frac{5}{2}$. From this graph it is seen that an approximate maximum value of y is 162.7 A closer approximation may be obtained by assigning to x a larger number of values between 2.0 and 2.4.

Exercises

1. Same as the illustrative problem above, but with the side of the square of tin equal to, (*a*) 10 inches; (*b*) 15 inches; (*c*) 17 inches; (*d*) 20 inches.

2. If a represents the side of the square of tin in exercise 1, and x the side of the square cut out, calculate the ratio x/a for each case.

3. Find the dimensions of the greatest rectangle that can be inscribed in a circle with radius 10.

140. The function $ax^2 + bx + c$. In § **31** it was shown that the function $ax^2 + bx + c$ has a maximum or a minimum value.

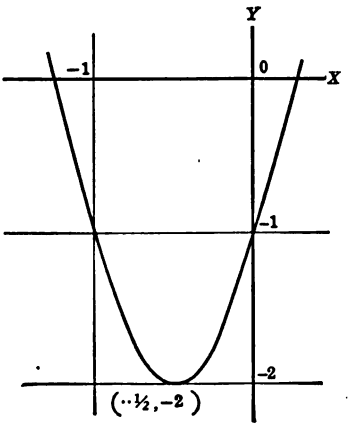


FIG. 112.

Thus, to illustrate, the function $y = 4x^2 + 8x - 2$ may be put in the form

$$y = 4x^2 + 8x + 1 - 2,$$

or

$$y = (2x + 1)^2 - 2,$$

or

$$y = 4(x + 1/2)^2 - 2.$$

The last equation shows that the vertex of the parabola is at the point $(-\frac{1}{2}, -2)$, and that the axis of the parabola extends up from the vertex, as is represented in Fig. 112. The function then

has a minimum value -2, which is the value corresponding to value $x = -\frac{1}{2}$.

Exercises

Find the maximum or minimum value for each of the following functions:

1. $x^2 + 2x + 1$.	**4.** $9x^2 + 3x - 2$.	**7.** $2x - x^2$.
2. $x^2 + 3x - 6$.	**5.** $7x^2 - 2x - 6$.	**8.** $3 - 5x - 2x^2$.
3. $4x^2 + 16x$.	**6.** $8x^2 - 3x - 7$.	**9.** $3 + 5x - 2x^2$.

10. If a body is thrown vertically upward with an initial speed of a feet per second, its height, h, in feet, at the end of t seconds is given by the equation $h = at - 16.1t^2$. To what height will the body rise if thrown with an initial speed of 32.2 feet per second? When will it reach this height?

141. Maxima and Minima Found by Limits. In some problems the maxima or minima values of a function may be found by a method illustrated by the following examples:

EXAMPLE 1: Find the maximum and minimum values, if any, of the functon

$$\frac{x^2 + 6}{2x + 1}.$$

To find the maximum and minimum values of the function is to find the ordinates of the turning points of the graph of the equation

$$y = \frac{x^2 + 6}{2x + 1}. \tag{1}$$

Upon solving for x the equation

$$x = y \pm \sqrt{(y - 2)(y + 3)} \tag{2}$$

is obtained.

If, in equation (2), values are assigned to y ($y = k$, where k is some constant) corresponding values of x are found; thus locating points upon the graph of the function. These values of x are given by

$$x = k \pm \sqrt{(k - 2)(k + 3)}. \tag{3}$$

The equation $y = k$ represents a straight line parallel to the x-axis, k units distant, and the values of x found from equation

(3) are the x-coördinates of the points of intersection of this straight line with the graph of equation (2).

The second factor, $k + 3$, under the radical of equation (3) is positive for all positive values of k, and for all negative values of k numerically less than 3, or positive for all values of k greater than -3. It is zero when k equals -3, and negative when k is less than -3. The first factor is positive when k is greater than 2, and zero when k equals 2, and negative when k is less than 2.

When both factors are positive, *i.e.*, when k is greater than 2 or less than -3, x is real, and the line $y = k$ cuts the graph of equation (2) in two real points. The x-coördinates of these points are the values of x found from equation (3).[1] When k

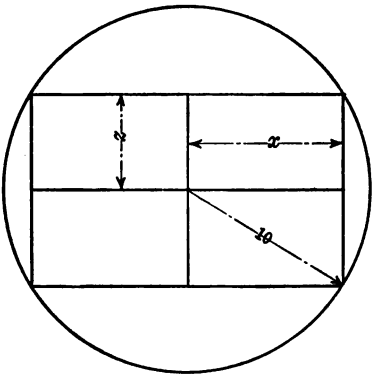


FIG. 113.

is greater than -3 and less than 2, x is imaginary, for then the quantity under the radical is negative. This shows that all lines of the system $y = k$ between the lines $y = 2$ and $y = -3$, fail to cut the curve. When $k = -3$ or 2 there is in each case but one value (or two equal values) of x. The lines $y = 2$ and $y = -3$ are tangent to the curve.

From the above discussion it is seen that the graph of equation (2) consists of two branches, one above and tangent to the

[1] The student should plot the graph of equation (2). Give to y in (2), or to k in (3), values ranging from -10 to $+10$.

line $y = 2$, the other below and tangent to the line $y = -3$. These two branches extend to infinity, one in the positive, the other in the negative y direction, for x is real for all positive k values greater than 2, and for all negative k values less than -3. The graph has two turning points, one with a y-coördinate 2, the other with a y-coördinate -3. The first is a minimum value, the second a maximum value of the given function. The minimum value corresponds to the value 2 for x, the maximum value corresponds to the value -3 for x.

EXAMPLE 2: Find the maximum rectangle that can be inscribed in a circle with radius 10.

In Fig. 113, let x be one-half the length of the base, and z one-half the length of the altitude. The area, y, is

$$y = 4xz. \tag{1}$$

$$z = \sqrt{100 - x^2}. \tag{2}$$

Substituting in equation (1)

$$y = 4x\sqrt{100 - x^2}. \tag{3}$$

Squaring equation (3)

$$y^2 = 16x^2(100 - x^2). \tag{4}$$

Solving (4) for x^2

$$x^2 = \frac{800 \pm \sqrt{(200 - y)(200 + y)}}{16}. \tag{5}$$

In equation (5) the quantity under the radical is positive if y is less than 200 and greater than -200, $(-200 < y < 200)$, and x^2 is real. Beyond these values for y, x^2 becomes imaginary. 200 then is the maximum value of y. When y equals 200, x^2 equals 50, or $x = \sqrt{50}$, and from equation (2), $z = \sqrt{50}$. The maximum rectangle is then a square with a side equal to $2\sqrt{50}$.

Exercises

1. A gutter with rectangular cross section is made from a strip of tin 30 inches wide. Find the width of the portion of tin bent up upon each side so that the carrying capacity of the gutter shall be a maximum.

HINT: Let y be the area of the cross section, and let x be the width of the tin bent up. Then $y = x(30 - 2x)$.

2. Find the length of the lever, exercise 6, **§ 91**, which will make the upward pull a minimum.

142. Another Method of Finding Maxima and Minima. The method considered here of finding the maxima and minima of a function depends upon a study of the slope of the tangent line drawn to the curve representing the function. In Fig. 109, a tangent line drawn to the curve at any point between P and A makes an acute angle with the positive direction of the axis of x. The slope of any such tangent line is then positive. Any tangent line drawn to the curve between the points A and B makes an obtuse angle with the positive direction of the axis of x. The slope of any such tangent line is then negative. The slope of any tangent lines drawn to the curve between the points B and C, and between the points D and E, is positive. The slope of any tangent lines drawn to the curve between the points C and D is negative. When the slope of the tangent line is positive, the curve ascends; when it is negative,

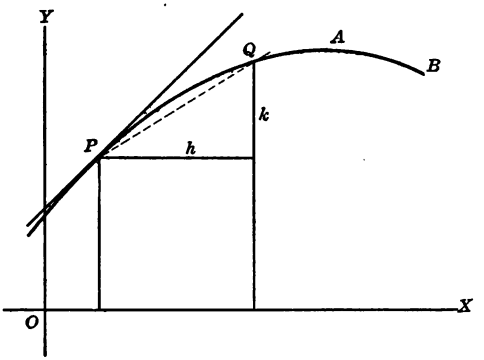


FIG. 114.

the curve descends. When the slope of the tangent line changes from positive to negative the curve changes from ascending to descending and possesses a high point. When the slope of the tangent line changes from negative to positive, the curve changes from descending to ascending and possesses a low point.

EXAMPLE 1: In § **139** the volume, y, of a box formed from a square piece of tin 13 inches on a side was found to be

$$y = (13 - 2x)^2x = 169x - 52x^2 + 4x^3. \qquad (1)$$

Let us suppose that the curve in Fig. 114 represents the graph of the function. Let P be any point on the curve whose coördinates

are x and y. Let Q be any other point on the curve whose coördinates are $x + h$ and $y + k$. Since these points are on the curve, their coördinates must satisfy equation (1), from which we obtain

$$y = 169x - 52x^2 + 4x^3 \tag{2}$$

and

$$y + k = 169(x + h) - 52(x + h)^2 + 4(x + h)^3. \tag{3}$$

Equation (3) reduces to

$$y + k = 169x + 169h - 52x^2 - 104hx - 52h^2 + 4x^3 + 12x^2h + 12xh^2 + 4h^3. \tag{4}$$

Subtracting equation (2) from equation (4) there results

$$k = 169h - 104hx + 52h^2 + 12hx^2 + 12xh^2 + 4h^3,$$

or

$$\frac{k}{h} = 169 - 104x + 52h + 12x^2 + 12xh + 4h^2. \tag{6}$$

k/h represents the slope of the secant line drawn through the points P and Q. If the point Q is taken nearer and nearer the point P, the secant line rotates about the point P and approaches the tangent line drawn to the curve at that point. The slope of the tangent line is then the limit of the slope of the secant line as the point Q approaches the point P. But, as Q approaches P, h approaches zero, and the right-hand side of equation (6) approaches $169 - 104x + 12x^2$. Therefore, the slope of the tangent line drawn to the curve representing the volume of the box is

$$12x^2 - 104x + 169, \text{ or } 12\left[x^2 - \frac{104}{12}x + \frac{169}{12}\right],$$

or

$$\textit{slope} = 12\left(x - \frac{13}{6}\right)\left(x - \frac{13}{2}\right). \tag{7}$$

Since the side of the square of tin is 13 inches, x, the side of the square cut from each corner, must be less than $6\frac{1}{2}$ inches. Thus, the last factor, $\left(x - \frac{13}{2}\right)$, of equation (7) is always negative. The second factor, $\left(x - \frac{13}{6}\right)$, is negative if x is less than $\frac{13}{6}$, and positive if x is greater than $\frac{13}{6}$. The expression representing the slope is then positive for values of x ranging from 0 to $\frac{13}{6}$, and

for these values of x the curve representing the function rises. For all values of x ranging from $\frac{13}{6}$ to 13 the slope is negative, and the curve descends. Thus, the function representing the volume of the box has a maximum value when $x = \frac{13}{6}$. To find the maximum volume substitute this value for x in equation (1), which then gives

$$y = 162.74.$$

EXAMPLE 2: The area of a rectangle inscribed in a circle, radius 10, is given by equation (3), example 2, § **141.**

$$y = 4x\sqrt{100 - x^2}, \tag{8}$$

or

$$\begin{aligned} y^2 &= 16x^2(100 - x^2) \\ &= 1600x^2 - 16x^4. \end{aligned} \tag{9}$$

A value of x which makes y a maximum will at the same time make y^2 a maximum. For brevity call $y^2 = u$, and equation (9) becomes

$$u = 1600x^2 - 16x^4. \tag{10}$$

As in the preceding example, let P and Q be any two points upon the curve, Fig. 114, which represents the relation between u and x. If x and u are the coördinates of the point P, and $x + h$ and $u + k$ those of the point Q,

$$u = 1600x^2 - 16x^4 \tag{11}$$

and

$$\begin{aligned} u + k &= 1600(x + h)^2 - 16(x + h)^4 \\ &= 1600x^2 + 3200hx + 1600h^2 - 16x^4 - 64hx^3 \\ &\qquad - 96h^2x^2 - 64h^3x - 16h^4. \end{aligned} \tag{12}$$

Upon subtracting equation (11) from equation (12) there results

$$k = 3200hx + 1600h^2 - 64hx^3 - 96h^2x^2 - 64h^3x - 16h^4, \tag{13}$$

or

$$\frac{k}{h} = 3200x + 1600h - 64x^3 - 96hx^2 - 64h^2x - 16h^3. \tag{14}$$

As h approaches zero, the right-hand side of equation (14) approaches $3200x - 64x^3$, or

$$slope = 64x(50 - x^2) = 64x(\sqrt{50} + x)(\sqrt{50} - x). \tag{15}$$

From the right-hand side of equation (15) it is seen that the slope is negative if x is greater than $\sqrt{50}$, and positive if x is positive and less than $\sqrt{50}$. The curve ascends for all values of x from 0 to $\sqrt{50}$, and descends for all values of x greater than $\sqrt{50}$. Thus, u has a maximum value when $x = \sqrt{50}$. When u is a maximum, y is a maximum; therefore the maximum rectangle is that for which $x = \sqrt{50}$. The remaining discussion of this problem has been given in **§ 141.**

EXAMPLE 3: Find the most economical proportion of a closed cylindrical can. Let z be the altitude, x the radius of the base, v the volume, and y the surface of the can. The problem is to find the ratio x/z which will make y a minimum for any constant value of v.

$$v = \pi z x^2 \tag{16}$$

$$y = 2\pi z x + 2\pi x^2 \tag{17}$$

Upon eliminating z between equations (16) and (17)

$$y = \frac{2v}{x} + 2\pi x^2. \tag{18}$$

$$y + k = \frac{2v}{x + h} + 2\pi(x + h)^2. \tag{19}$$

Subtracting equation (18) from equation (19)

$$k = \frac{2v}{x + h} - \frac{2v}{x} + 2\pi(2xh + h^2),$$

or

$$k = \frac{-2vh}{x(x + h)} + 2\pi(2xh + h^2),$$

or

$$\frac{k}{h} = 2\pi(2x + h) - \frac{2v}{x(x + h)}. \tag{20}$$

As h approaches zero, the right-hand side of equation (20) approaches $4\pi x - \frac{2v}{x^2}$. Then

$$slope = 4\pi x - \frac{2v}{x^2},$$

or

$$slope = \frac{4\pi}{x^2}\left(x^3 - \frac{v}{2\pi}\right). \tag{21}$$

If x is less than $\sqrt[3]{\frac{v}{2\pi}}$ the quantity within the parentheses of equation (21) is negative. If x is greater than $\sqrt[3]{\frac{v}{2\pi}}$ this quantity is positive. For all positive values of x less than $\sqrt[3]{\frac{v}{2\pi}}$ the slope is negative, and the curve representing y is descending; for values of x greater than $\sqrt[3]{\frac{v}{2\pi}}$ the curve is ascending. Thus, the surface is a minimum when $x = \sqrt[3]{\frac{v}{2\pi}}$. From equation (16) $z = \sqrt[3]{\frac{4v}{\pi}}$, whence $\frac{x}{z} = \frac{1}{2}$. Since the v does not appear in this equation, the ratio of x to z is constant, $\frac{1}{2}$, for all values of v. The most economical proportions are such that the altitude is equal to the diameter of the base.

The method used in the three above cases is applicable to any problem where the functional relation between the two variable quantities x and y is given by means of an equation.

Exercises

1. Find the most economical dimensions of a pan with square base and vertical sides, if it is to hold 5 gallons (1 gallon = 231 cubic inches).

2. Find the most economical proportions for a cylindrical can, open at the top, if it is to hold 1 quart (1 quart = $57\frac{3}{4}$ cubic inches).

3. What is the ratio of the length of the legs of a right triangle when its area is a maximum, if the hypotenuse is a constant, 10?

4. The strength, y, of a rectangular beam varies as the product of its breadth, z, by the square of its depth, x; find the dimensions of the strongest beam which can be cut from a log 20 inches in diameter.

CHAPTER XIII

[EMPIRICAL EQUATIONS]

143. Empirical Equation Defined. A series of readings connecting two or more variable numbers should always be plotted upon rectangular coördinate paper (squared paper), thus representing to the eye the variation of one of the numbers with respect to the other.

In Table VI, there are given seven readings of the velocity of water through a sample of sand. These seven velocities correspond each to a measured hydraulic gradient. This set of readings when plotted upon squared paper is shown in Fig. 115.

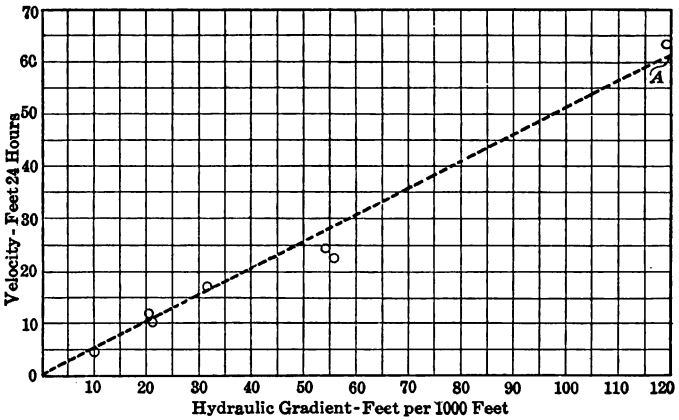


FIG. 115.

The plotted points do not lie upon a smooth curve. The reason for this is that there are errors of observation—errors both in the determinations of the velocities and in the hydraulic gradients. If there had been no errors of observation and if all pairs of readings had been taken under exactly the same conditions, the plotted

points would have fallen upon a smooth curve. This smooth curve which we are trying to locate by the plotted points represents graphically the law of nature which we are trying to determine or study, viz., the law governing the flow of water through this particular sample of sand, relative to the hydraulic gradient.

From Fig. 115 it is apparent that the plotted points do not justify the assumption of a locus other than a straight line along which the points would fall if there were no errors of observation. The true relation connecting the variable numbers measured may not be the linear law, but with the data at hand this is the only law which can be assumed as an approximation to the law of nature. With the aid of a transparent triangle the line $0A$

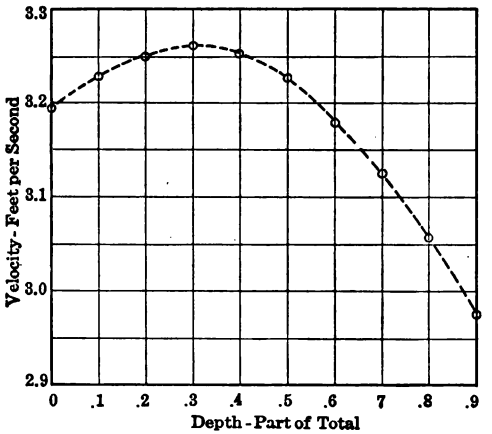


FIG. 116.

is drawn among the plotted points. The position of the line is located by the eye as the probable line upon which the points would fall if the true law were linear and if there were no errors of observation. The equation of this line is then $y = ax + b$, where x represents the hydraulic gradient and y the velocity. Since the line passes through the origin, b is zero. a is found to be 0.51. The equation of the line becomes $y = 0.51x$, and is called an empirical equation connecting velocity and hydraulic gradient.

Frequently it is desirable to express the relation existing between variable numbers in the form of an equation, even though this equation represents only an approximation to the true law. Such an approximate equation when found by means of experimental readings is called an **empirical equation.**

As a second illustration of a method of finding the empirical equation, consider the plotted points in Fig. 116. From this figure it is seen that the law connecting velocity and depth is not linear, but appears to be parabolic. The vertex of the parabola is at, or nearly at, the point (0.29, 3.262)[1]. If the origin be moved to this point the readings (which are given in Table IX) become:

x'	y'	x'	y'
− 0.29	− 0.067	0.21	− 0.034
− 0.19	− 0.032	0.31	− 0.082
− 0.09	− 0.009	0.41	− 0.135
0.01	− 0.001	0.51	− 0.203
0.11	− 0.010	0.61	− 0.286

and the equation connecting x' and y' is $y' = \alpha(x')^2$.

If $(x')^2$ be replaced by X the equation becomes

$$y' = \alpha X \tag{1}$$

Hence if y''s are plotted against X's, the squares of the x''s, a straight line will be obtained if the curve in Fig. 116 is a parabola with its vertex at the point (0.29, 3.262). These plotted points are represented in Fig. 117. The points indicated by circles correspond to positive values of x', while those indicated by dots correspond to negative values. A straight line drawn among the points with the aid of a transparent triangle gives the equation

$$y' = - 0.77X. \tag{2}$$

Since $X = (x')^2$,

$$y' = - 0.77(x')^2. \tag{3}$$

Since

$$x' = x - 0.29$$

and

$$y' = y - 3.262,$$

[1] From Fig. 116 it may at first appear that the vertex of the parabola is on the line $x = 0.3$; but by noting the vertical positions of the points plotted equidistant from this line it is seen that the line of symmetry of the curve is slightly to its left.

equation (3) becomes

$$y - 3.262 = -0.77(x - 0.29)^2,$$

or

$$y = -0.77(x - 0.29)^2 + 3.262, \tag{4}$$

or

$$y = -0.77\,x^2 + 0.447\,x + 3.197, \tag{5}$$

an empirical equation connecting velocity with depth.[1]

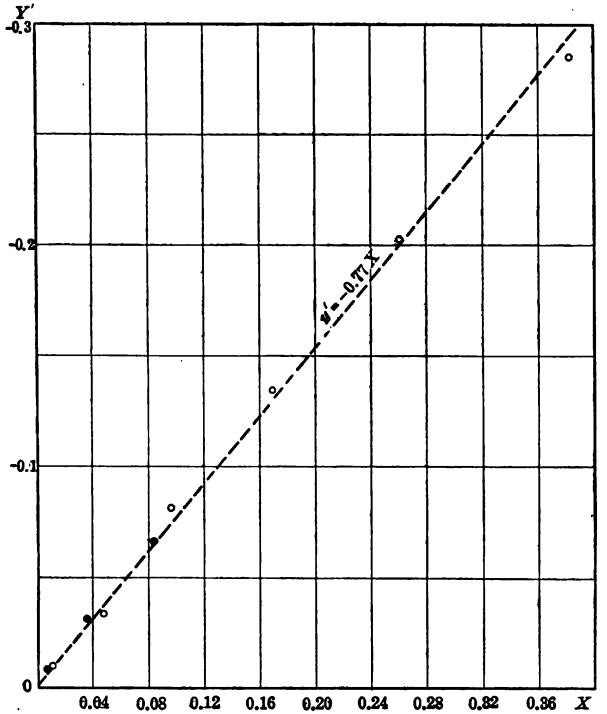


FIG. 117.

In the first illustration above the empirical equation is of the form $y = ax + b$, in the second illustration it is of the form

[1] Compare equation (4) with the equation given in exercise 4, § 31.

$y = ax^2 + bx + c$. These equations are each a special case of the more general equation

$$y = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n, \qquad (6)$$

where $a_0, a_1, a_2, \ldots, a_n$ are coefficients to be determined and where n is a positive integer. To determine the $n + 1$ coefficients of equation (6), $n + 1$ equations are required. Hence the curve corresponding to (6) may be made to pass through $(n + 1)$ points of the curve drawn among the plotted points. Empirical equations of the form (6) are undesirable if more than three or four terms are required.

Exercises

1. From the data given in Table VII, find empirical equations connecting yield and head, for the 8-inch and 14-inch wells.

2. From the data given in Table VIII, find an empirical equation connecting T and W.

144. Coefficients Determined by the Method of Least Squares.[1] In Fig. 115 a linear law was assumed connecting velocity and hydraulic gradient. A straight line was drawn among the points and the values of the coefficients a and b were determined from this line. If the plotted points had been so scattered that it would have been difficult to estimate by eye the position of the line, the coefficients a and b could have been determined by the method of least squares. This method will be illustrated by using the two illustrations of the preceding section. In the first illustration the equation is assumed to be $y - ax - b = 0$. Upon substituting the seven pairs of values given in Table VI there results:

$$\begin{aligned}
16.9 - \ \ 31.9\,a - b &= 0 \\
11.4 - \ \ 20.8\,a - b &= 0 \\
24.5 - \ \ 54.1\,a - b &= 0 \\
\ \ 4.3 - \ \ 10.0\,a - b &= 0 \\
10.1 - \ \ 21.0\,a - b &= 0 \\
58.8 - 119.1\,a - b &= 0 \\
22.7 - \ \ 55.8\,a - b &= 0
\end{aligned}$$

[1] The student will verify the numerical work of this section.

Here are seven equations from which to solve for a and b. This can, of course, not be done, since the number of equations is greater than the number of unknowns. The values of a and b, which are regarded as the best approximations, are found as follows: First multiply each member of each equation by the coefficient of a in that equation and add the resulting seven equations. In the illustration this gives

$$10627.52 - 22216.51a - 312.7b = 0. \qquad (1)$$

Next multiply each member of each equation by the coefficient of b in that equation and add. In the illustration this gives

$$148.7 - 312.7a - 7b = 0. \qquad (2)$$

Next solve equations (1) and (2) for a and b. This gives in the illustration

$$a = 0.48$$

and

$$b = -0.2$$

Thus, the equation of the straight line drawn among the points determined by this method is

$$y = 0.48x - 0.2$$

The equation found by the method of the preceding section is

$$y = 0.51x$$

In the second illustration (See Fig. 116) the empirical equation was assumed to be of the form $y - \alpha x^2 - \beta x - \gamma = 0$. Upon substituting the ten pairs of values given in Table IX there results;

$$\begin{aligned}
3.195 - 0.00\,\alpha - 0.0\,\beta - \gamma &= 0 \\
3.230 - 0.01\,\alpha - 0.1\,\beta - \gamma &= 0 \\
3.253 - 0.04\,\alpha - 0.2\,\beta - \gamma &= 0 \\
3.261 - 0.09\,\alpha - 0.3\,\beta - \gamma &= 0 \\
3.252 - 0.16\,\alpha - 0.4\,\beta - \gamma &= 0 \\
3.228 - 0.25\,\alpha - 0.5\,\beta - \gamma &= 0 \\
3.181 - 0.36\,\alpha - 0.6\,\beta - \gamma &= 0 \\
3.127 - 0.49\,\alpha - 0.7\,\beta - \gamma &= 0 \\
3.059 - 0.64\,\alpha - 0.8\,\beta - \gamma &= 0 \\
2.976 - 0.81\,\alpha - 0.9\,\beta - \gamma &= 0
\end{aligned}$$

To find the "best values" for α, β and γ from these equations, first multiply each member of each equation by the coefficient of α in that equation and add the resulting ten equations. There results

$$8.8289 - 1.5333\alpha - 2.025\beta - 2.85\gamma = 0. \quad (3)$$

Next multiply each member of each equation by the coefficient of β in that equation and add the resulting equations. There results

$$14.09 - 2.025\alpha - 2.85\beta - 4.5\gamma = 0. \quad (4)$$

Next multiply each member of each equation by the coefficient of γ in that equation and add the resulting ten equations. There results

$$31.762 - 2.85\alpha - 4.5\beta - 10\gamma = 0. \quad (5)$$

Next solve equations (3), (4), and (5) for α, β, and γ, and obtain

$$\alpha = -0.81$$
$$\beta = 0.48$$
$$\gamma = 3.19$$

Hence an empirical equation found by this method is

$$y = -0.81x^2 + 0.48x + 3.19$$

This equation places the vertex of the parabola at the point (0.296, 3.26).

These two illustrations show that a large amount of numerical work is required to find the coefficients by the method of least squares. If the plotted points fall upon, or nearly upon, a smooth curve the methods of the preceding section should be used.[1]

145. Logarithmic Paper. The student will read **§ 73** again. In Fig. 118 the square $ADCB$ is one unit on a side. At the bottom and at the left-hand side of this square are logarithmic scales, each ranging from 1 to 10. Thus on the lower scale, AD, the numbers are placed opposite their logarithms measured on the uniform scale CB. For example, 1 is placed opposite 0; 2 opposite 0.301; 3 opposite 0.477; 4 opposite 0.602; 5 opposite 0.699; 6 opposite 0.778; 7 opposite 0.845; 8 opposite 0.903; 9 opposite 0.954, and 10 opposite 1. The distance from A to the position of any number on the scale AD is equal to the logarithm of that number.

[1] The theory upon which the method of least squares is based may be found in texts on "Least Squares."

Through the points of division of the AD and AC scales, vertical and horizontal lines are drawn forming what is called **logarithmic coördinate paper.**[1] To illustrate the use of logarithmic coördinate

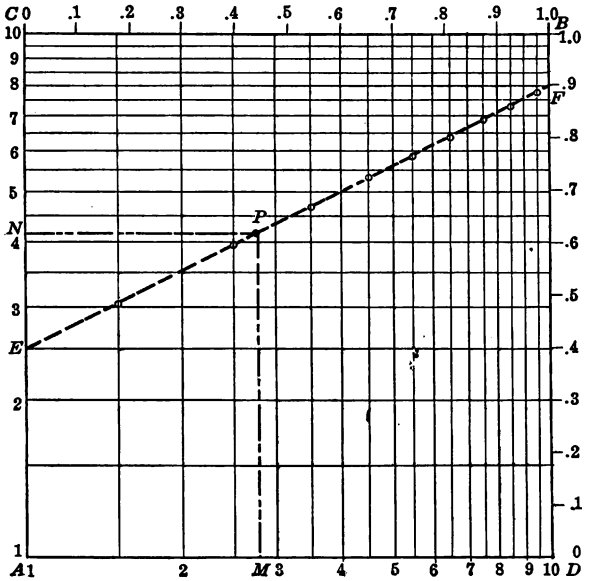


FIG. 118.—A single square (reduced) of logarithmic paper.

paper for determining empirical equations, consider the data given in the accompanying table. These data are plotted upon squared paper in Fig. 119, and upon logarithmic paper in Fig. 118.

x	y	x	y
1.5	3.05	6.5	6.40
2.5	3.92	7.5	6.85
3.5	4.65	8.5	7.25
4.5	5.30	9.5	7.70
5.5	5.82	.....	.....

[1]For the work in this chapter the student should provide himself with Single Logarithmic cross-section paper, form *M*4, Semi-logarithmic coördinate paper, form *M*5, and Multiple Logarithmic coördinate paper, form *M*6.

In Fig. 118 the plotted points lie upon a straight line EF. If we imagine the logarithmic rulings to be removed and rulings corresponding to the scales CB and BD to be drawn, the line EF would then appear upon a sheet of squared paper with the origin at A and the point (1, 1) at B. The equation of the line EF is then $Y = aX + b$, where X and Y are the rectangular coördinates of any point, P, on the line.[1] a, the slope of the line, is found by measure-

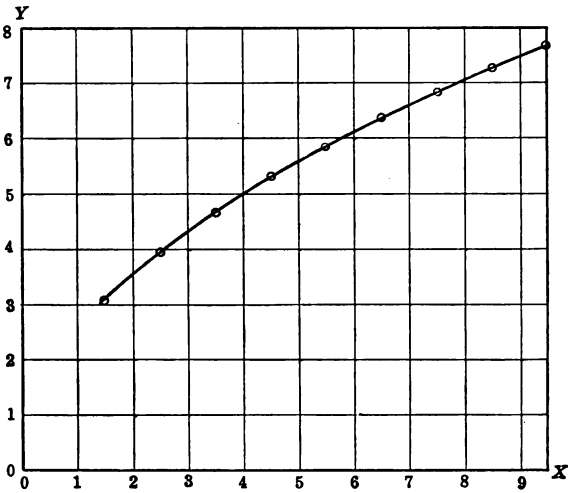


FIG. 119.

ment,[2] to be $\frac{1}{2}$. b, the y-intercept AE, is found from the scale DB to be equal to 0.398, which, from the scale AC, is seen to be equal to log 2.5. The equation of the straight line EF is then

$$Y = \tfrac{1}{2}X + \log 2.5 \qquad (1)$$

In this equation Y is the distance, PM, of any point, P, from the line AD; but this distance is the logarithm of y. Similarly, X, the distance PN, is the logarithm of x. Hence equation (1) becomes

$$\log y = \tfrac{1}{2}\log x + \log 2.5,$$

[1] X and Y are used to represent the rectangular coördinates of a point in contradistinction to the x and y when the logarithmic scales are used.

[2] The student is supposed to reproduce Fig. 118 on a sheet of logarithmic paper, form *M4*.

or

$$\log y = \log x^{\frac{1}{2}} + \log 2.5,$$

or

$$\log y = \log (2.5x^{\frac{1}{2}}),$$

or

$$y = 2.5x^{\frac{1}{2}}, \tag{2}$$

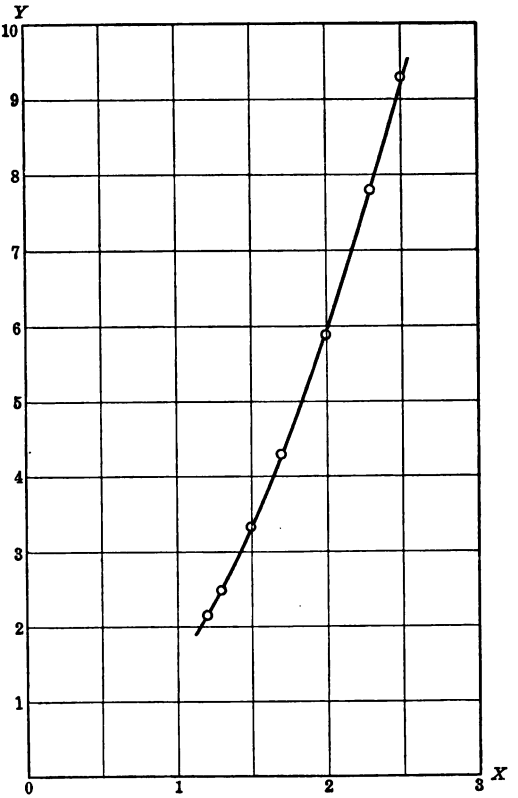


FIG. 120.

an empirical equation connecting the x and y values given in the above table.

As a second illustration, consider the data given in the accompanying table. These data are shown plotted upon squared paper in Fig. 120, and upon logarithmic paper in Fig. 121. Upon the

x	y	x	y
1.2	2.15	2.0	5.90
1.3	2.50	2.3	7.80
1.5	3.35	2.5	9.30
1.7	4.30	.....	.....

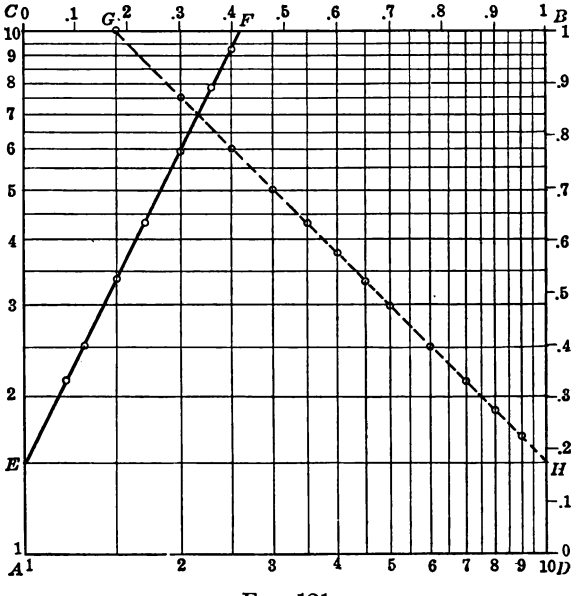


FIG. 121.

logarithmic paper the curve is a straight line. Hence its equation is $Y = aX + b$. The a is found by measurement to be 2. The b is seen to be log 1.5. Then

$$Y = 2X + \log 1.5,$$

or

$$\log y = 2 \log x + \log 1.5,$$

or

$$\log y = \log x^2 + \log 1.5,$$

or

$$\log y = \log 1.5x^2,$$

or

$$y = 1.5x^2,$$

an empirical equation connecting the x- and y-values of the above table.

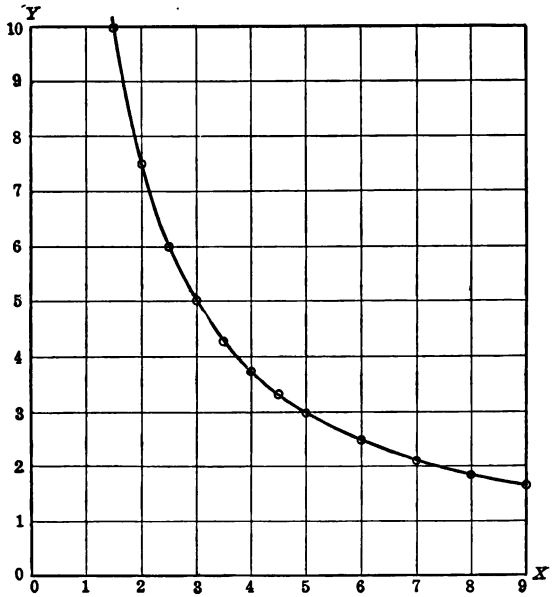


FIG. 122.

As a third illustration consider the data given in the accompanying table. These data are shown plotted upon squared paper in Fig. 122 and upon logarithmic paper in Fig. 121. The plotted points in Fig. 121 lie upon the straight line GH, whose equation is $Y = aX + b$. a is found to be -1, and b the log 1.5. Hence

$$Y = -X + \log 1.5,$$

x	y	x	y
1.5	10.0	4.5	3.30
2.0	7.5	5.0	2.98
2.5	6.0	6.0	2.49
3.0	5.0	7.0	2.12
3.5	4.25	8.0	1.87
4.0	3.73	9.0	1.65

or

$$\log y = -\log x + \log 1.5,$$

or

$$\log y = \log \frac{1}{x} + \log 1.5,$$

or

$$\log y = \log \frac{1.5}{x},$$

or

$$y = \frac{1.5}{x},$$

an empirical equation connecting the x and y values of the table.

From the above illustrations it is seen that if plotted points upon logarithmic paper determine a straight line, an empirical equation connecting the numbers plotted is of the form

$$y = Kx^n,$$

where n is the slope of the line as it appears on the logarithmic paper, and where K is the reading on the logarithmic scale of the point of intersection of the line with the vertical line passing through the point (1, 1). Conversely, any equation of the form $y = Kx^n$ plotted upon logarithmic paper gives a straight line. For, by taking the logarithm of each member of the equation there results

$$\log y = n \log x + \log K,$$

which is of the form $Y = nX + b$.

From the above illustrations it is seen that the uniform scales at the top and right-hand edge of the logarithmic paper are not used in the actual determination of the constants in the empirical equation. These scales are placed there for instructional purposes only.

Exercises

1. Through the lower left-hand corner of a sheet of logarithmic paper, form $M4$, draw lines making angles of 30°, 45°, and 60° with the horizontal. Find the equations connecting the x and y for each of these lines.

2. Through the upper right-hand corner of a sheet of logarithmic paper, form $M4$, draw lines making angles of 30°, 45°, and 60° with the horizontal. Find equations connecting the x and y for each of these lines.

3. Through the upper left-hand corner of a sheet of logarithmic paper, form $M4$, draw lines making angles of 30°, 45°, and 60° with the vertical. Find equations connecting the x and y for each of these lines.

4. Through the lower right-hand corner of a sheet of logarithmic paper, form $M4$, draw lines making angles of 30°, 45°, and 60° with the horizontal. Find equations connecting x and y for each of these lines.

5. From a sheet of logarithmic paper, form $M4$, find to three decimal places the logarithm of (*a*) 2.1; (*b*) 3.6; (*c*) 4.8; (*d*) 5.2; (*e*) 7.21; (*f*) 9.3; (*g*) 9.8.

6. Find an empirical equation connecting x and y of the accompanying table.

x	y	x	y
1.5	3.55	4.0	1.59
2.0	2.80	5.0	1.22
2.5	2.34	6.0	1.14
3.0	2.02	.....	.....

7. Draw lines upon a sheet of logarithmic paper, form $M4$, representing

(*a*) $y = x^2$.
(*b*) $y = x^{1/2}$.
(*c*) $y = 2x^3$.
(*d*) $y = 2x^{1/2}$.
(*e*) $y = \frac{1}{x}$.
(*f*) $y = \frac{2}{x}$
(*g*) $y = 3x^{3/4}$.
(*h*) $y = 8x^{-3/4}$.
(*i*) $y = 1.5x^{2/3}$.
(*j*) $y = 9.2x^{-2/3}$.

146. Multiple Logarithmic Paper. In the preceding section logarithmic paper was used (form $M4$) in which each of the two

logarithmic scales extended from 1 to 10, or the corresponding uniform scales extended from 0 to 1. It is apparent that if numbers falling beyond these limits were to be plotted, the coördinate paper would have to be extended. Fig. 123 represents a sheet of

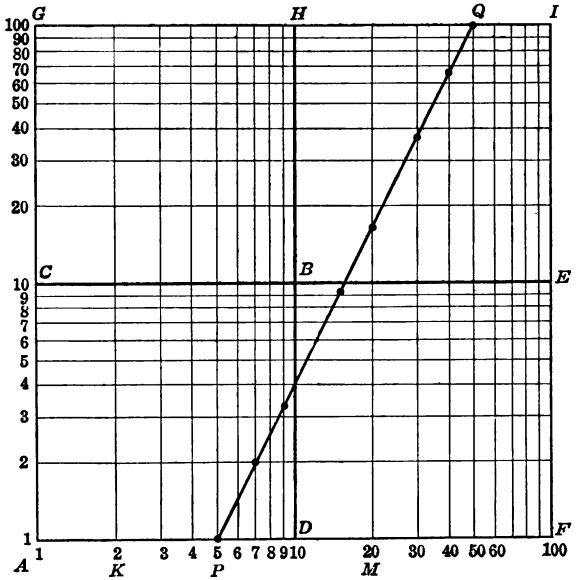


FIG. 123.—Four squares of multiple logarithmic paper (reduced).

coördinate paper in which each logarithmic scale extends from 1 to 100. The square $ADBC$ is the same as the square $ADBC$ of Fig. 118, or of form $M4$.

Since

$$\begin{aligned} \log 20 &= \log 10 + \log 2 \\ &= 1 + \log 2 \\ &= AD + AK \\ &= AM \\ DM &= AK \end{aligned}$$

Thus the vertical line marked 20 is at the same distance from D as the line marked 2 is from A. Similarly the vertical lines

marked 30, 40, . . ., 90, and 100 are at the same distances from D, respectively, as the lines marked 3, 4, . . ., 9, and 10 are from the point A. Similarly the portion CG of the vertical scale is the same as the portion AC excepting that the numbers attached to the upper portion are ten times the numbers attached to the corresponding points on the lower portion. By the addition of squares like $ADBC$ the logarithmic paper may be extended over any range of values on the horizontal or vertical scales.

It is apparent from an examination of Fig. 123 that the point (1, 1) (the origin) may be taken at any corner of a square. Thus, if B is taken as origin, the point A is (0.1, 0.1) and I the point (10, 10); if H is taken as the origin, A is the point (0.1, 0.01) and I the point (10, 1). In this work it is supposed that the student has a supply of multiple logarithmic coördinate paper, form $M6$.

It is apparent that with logarithmic coördinate paper the origin, *i.e.*, the point (1, 1), may be taken only at the corners of the squares, while with squared paper the origin, *i.e.*, the point (0, 0), may be taken at any convenient point.

To illustrate the use of multiple logarithmic paper consider the data given in the accompanying table. The plotted points corresponding to these numbers are shown in Fig. 123. In plotting these points the point B is taken as the origin. The slope of the line PQ is 2 and the intercept is log 0.4. Hence the equation connecting the x- and y-values of the table is $y = 0.4x^2$.

x	y	x	y
0.5	0.10	2.0	1.64
0.7	0.20	3.0	3.70
0.9	0.33	4.0	6.50
1.5	0.92	5.0	10.00

Exercises

1. Find an empirical equation between x' and y', using the numbers given in the table on page 266.

2. Find an empirical equation connecting W and T, using the data given in Table VIII.

HINT: First move the zero to correspond to 4° centigrade. Let T'

be the temperature referred to this new zero. Find an equation between W and T', and then replace T' by $T - 4$.

3. Find an empirical equation connecting I and W, from the data given in Table XI.

147. Translating the Curve on Logarithmic Paper. Consider the data given in the accompanying table. These data plotted upon multiple logarithmic paper are represented by small circles in Fig. 124. The plotted points do not lie upon a straight

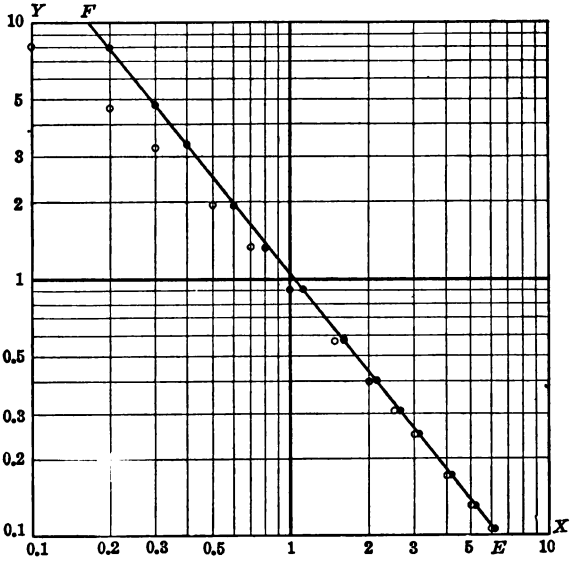


FIG. 124.

line, but if they are translated one unit to the right they fall upon the straight line EF. Moving the points one unit to the right is equivalent to plotting y against $x + 1$. Let $x + 1$ be represented by x'. From the line EF, the equation connecting y and x' is found to be

$$y = 1.05(x')^{-\frac{1}{5}}. \qquad (1)$$

Since $x' = x - 1$, equation (1) gives

$$y = 1.05(x - 1)^{-\frac{1}{5}},$$

an empirical equation connecting x and y.

x	y	x	y
0.1	8.00	2.0	0.40
0.2	4.75	2.5	0.31
0.3	3.30	3.0	0.25
0.5	1.95	4.0	0.17
0.7	1.35	5.0	0.13
1.0	0.90	6.0	0.105
1.5	0.57	.....	.....

Frequently it may happen that by translating the curve in the x direction or in the y direction the plotted points upon logarithmic paper will lie upon a straight line.

Exercise

1. By plotting upon logarithmic paper, form $M6$, find an empirical equation connecting volume and pressure, from the data given in Table X.

148. Equations Plotted upon Logarithmic Paper. If a large number of readings, say x, are to be substituted in a formula giving a second number, say y, it may be desirable to plot a curve representing the equation connecting x and y, and from this curve to scale off the values of y corresponding to the readings of x. If this equation plotted upon logarithmic paper gives a straight line it is an easy matter to construct the curve. Thus, as an illustration, consider the formula

$$Q = 3.37LH^{3/2}, \tag{1}$$

giving the discharge Q, in cubic feet per second (second-feet), over a trapezoidal weir. L is the length, in feet, of the crest of the weir, and H is the depth, in feet, of the water on the crest. This formula for various values of L will give on logarithmic paper a series of parallel lines of slope $\frac{3}{2}$.

Lines for L equal to 1, 2, 3, 4, and 5 are given in Fig. 125.

If only one curve of the family is to be drawn, as the line for $L = 2$, it is advisable to use a single square of logarithmic paper, form $M4$. From Fig. 125 it is seen that the line $L = 2$ crosses six squares. Instead of using these six squares a single square may be traversed six times as shown in Fig. 126. The segments of the line

in Fig. 126 are lettered the same as in Fig. 125. The segment AB is not shown in Fig. 126 as it coincides with the segment $F2$.

The terms tenths, units, and tens written above the lines indicate the denomination of the figures on the vertical scale. Similarly

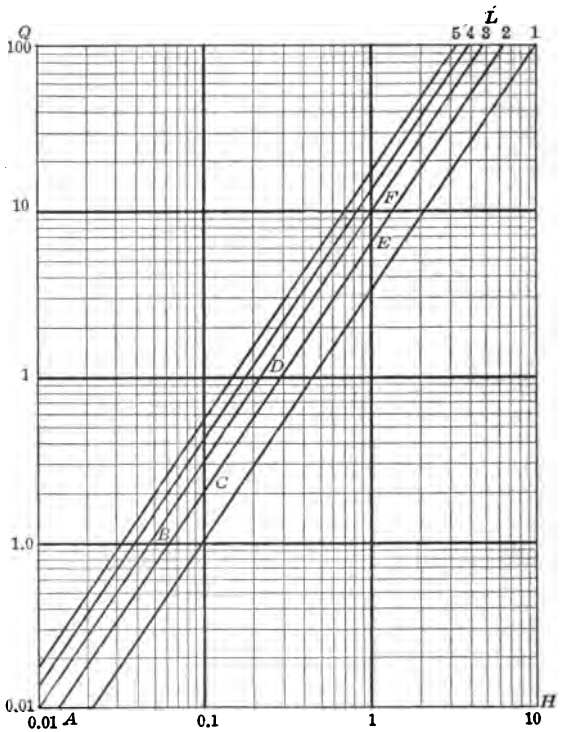


FIG. 125.—Curves for $Q = 3.37LH^{3/2}$ plotted upon logarithmic paper.

the terms hundredths, tenths, etc., written below the lines indicate the denomination of the figures on the horizontal scale. Thus, to illustrate, if the reading of H is 0.7, the line segment DE is used, and from it and the vertical scale the discharge, Q, is found to be 3.95 second feet.

Exercises

1. Upon a sheet of logarithmic paper, form $M6$, draw a series of curves for

$$Q = 3.37LH^{3/2}$$

for $L = 1, 2, 3, 4$, and 5.

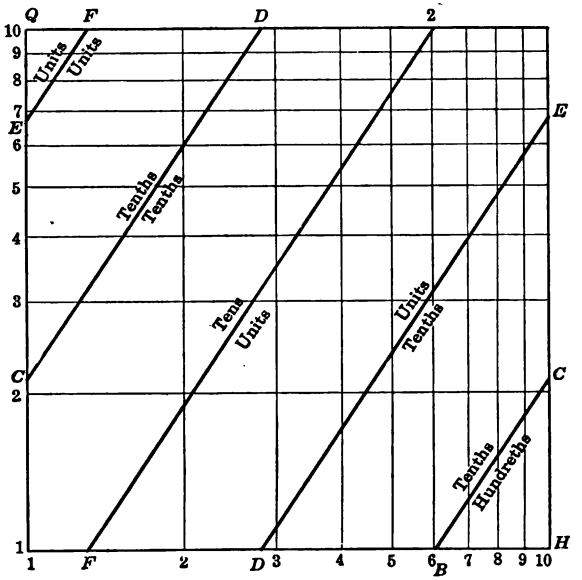


FIG. 126.—Curve for $Q = 3.37 \cdot 2H^{3/2}$ plotted upon one square of logarithmic paper.

2. Upon a sheet of logarithmic paper, form $M4$, draw a series of line segments for

$$Q = 3.37LH^{3/2}$$

if $L = 3$, and if H ranges from 0.1 to 10.

149. Semi-logarithmic Paper. Semi-logarithmic paper, form $M5$, is coördinate paper ruled logarithmically in one direction and uniformly in the other.

The data given in the accompanying table, plotted upon semi-logarithmic paper, is shown in Fig. 127.

x	y
0.2	3.18
0.4	3.96
0.6	5.00
0.8	6.30

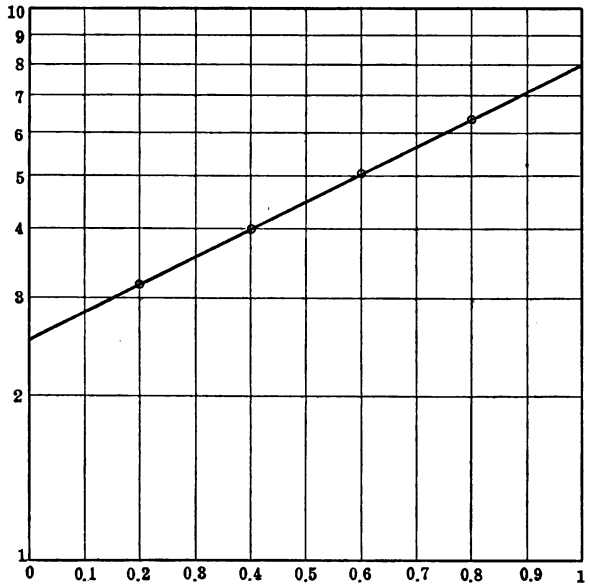


FIG. 127.—Semi-logarithmic paper (reduced).

The equation of this straight line is

$$Y = \tfrac{1}{2}x + \log 2.51,$$

or

$$\log y = \tfrac{1}{2}x + \log 2.51;$$

or

$$\log \frac{y}{2.51} = \frac{x}{2},$$

or

$$\frac{y}{2.51} = 10,x/2$$

or

$$y = 2.51(10)x/2,$$

an empirical equation connecting the x- and y-values of the table.

Exercises

1. Find an empirical equation connecting the x- and the y-values given in the accompanying table.

x	y
0.2	5.8
0.4	4.4
0.6	3.4
0.8	2.6

APPENDIX

GREEK ALPHABET

Letters	Names	Letters	Names	Letters	Names
A α	Alpha	I ι	Iota	P ρ	Rho
B β	Beta	K κ	Kappa	Σ σ, s	Sigma
Γ γ	Gamma	Λ λ	Lambda	T τ	Tau
Δ δ	Delta	M μ	Mu	Υ υ	Upsilon
E ϵ	Epsilon	N ν	Nu	Φ ϕ, φ	Phi
Z ζ	Zeta	Ξ ξ	Xi	X χ	Chi
H η	Eta	O o	Omicron	Ψ ψ	Psi
Θ θ, ϑ	Theta	Π π	Pi	Ω ω	Omega

SYMBOLS

$+$	plus.
$-$	minus.
$\pm$	plus or minus.
$\mp$	minus or plus.
$=$	equals.
$a \times b$, or $a \cdot b$, or ab	a times b.
$a \div b$, or $a:b$, or $\frac{a}{b}$	a divided by b.
( ), [ ], { }	parentheses, brackets, braces.
a^2	a square.
a^3	a cube
a^4	a to the fourth power.
$a^n = a\,a\,a\,\ldots\,a$	a to the nth, or, a nth (n a's multiplied together, if n is a positive integer).
$a^{1/2} = \sqrt{a}$	square root of a.
$a^{1/3} = \sqrt[3]{a}$	cube root of a.
$a^{1/r} = \sqrt[r]{a}$	rth root of a.
$a^{p/q} = \sqrt[q]{a^p} = \left[\sqrt[q]{a}\right]^p$	
$a^0 = 1$	
$a^{-s} = \frac{1}{a^s}$	

a', a'', a''', . . ., a^n	(accents) read, respectively, "a prime," "a second," "a third," etc.
a_0, a_1, a_2, . . ., a_n	(subscripts) read, respectively, "a sub zero," "a sub one," "a sub two," etc.
$\neq$	is not equal to.
$>$	is greater than.
$<$	is less than.
$\geqslant$	is equal to or greater than.
$\leqslant$	is equal to or less than.
$\angle$	angle.
$\lfloor\underline{n}$	(read, "factorial n") means the product of all the integral numbers from 1 up to and including n. Thus: $\lfloor\underline{5} = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5$.
$n!$	($\lfloor\underline{n}$ is sometimes written $n!$).
$\lvert a \rvert$	the numerical, or absolute, value of a.
$f(x)$; $\phi(x)$; $F(x)$; $\psi(x)$, etc.	function of x.
$\log_a x$	the logarithm of x to the base a.
$\log x$	the logarithm of x to the base 10.
Σ	the sum of such terms as
$\sum_{i=\alpha}^{i=\beta}$	the sum of such terms as, . . ., as i varies from α to β.
π	the ratio of the length of the circumference of a circle to the length of its diameter. An approximate value of π is 22/7. A better approximation is 3.1416.
e	the base of the natural (or hyperbolic) logarithmic system. e = 2.71828 approximately.
${}_nP_r = \frac{\lfloor\underline{n}}{\lfloor\underline{n-r}}$	the number of permutations of n things taken r at a time.
$P_n = \lfloor\underline{n}$	the number of permutations of n things taken all at a time.

$_nC_r = \frac{\lfloor n}{\lfloor n \, \lfloor n - r}$	the number of combinations of n things taken r at a time.
∞	infinite.
$\lim$	limit.
$\doteq$	approaches.
$\lim_{x \doteq a} Y$	the limit of Y as x approaches a.

FORMULAS

Algebra

1. $a^m a^n = a^{m+n}$.

2. $a^m \div a^n = a^{m-n}$.

3. $[a^m]^n = a^{mn}$.

4. $\left[\frac{a}{b}\right]^m = \frac{a^m}{b^m}$.

5. $(-a)(b) = -(ab)$.

6. $(-a)(-b) = ab$.

7. $(a \pm b)^2 = a^2 \pm 2ab + b^2$.

8. $(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3$.

9. $(a \pm b)^4 = a^4 \pm 4a^3b + 6a^2b^2 \pm 4ab^3 + b^4$.

10. $(a \pm b)^n = a^n \pm na^{n-1}b + \frac{n(n-1)}{\lfloor 2} a^{n-2}b^2 + \frac{n(n-1)(n-2)}{\lfloor 3} a^{n-3}b^3 + \cdots$

11. $a^2 \pm 2ab + b^2 = (a \pm b)^2$.

12. $a^2 - b^2 = (a + b)(a - b)$.

13. $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.

14. $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.

15. If $ax^2 + bx + c = 0$, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

16. $\log_a (xy) = \log_a x + \log_a y$.

17. $\log_a \frac{x}{y} = \log_a x - \log_a y$.

18. $\log_a x^n = n \log_a x$.

19. $\log_a \frac{1}{x} = -\log_a x$.

20. $\log_a x = (\log_b x)(\log_a b)$.

21. $\log_a b = \frac{1}{\log_b a}$.

22. $\log_e x = \log_e 10 \log_{10} x = 2.3026 \log_{10} x$.

23. $\log_{10} x = \log_{10} e \log_e x = 0.4343 \log_e x$.

24. $a + [a + d] + [a + 2d] + \cdots + \Big[a + (n-1)d\Big]$
$$= \frac{n}{2}\Big[2a + (n-1)d\Big].$$

25. $a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} = \dfrac{a(1 - r^n)}{1 - r}.$

26. $a + ar + ar^2 + ar^3 + \cdots \text{ to } \infty = \dfrac{a}{1 - r}$, if $|r| < 1$.

27. $1 + 2 + 3 + 4 + \cdots + n = \dfrac{n}{2}(n + 1).$

28. $1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2 = \dfrac{n}{6}(n + 1)(2n + 1).$

29. $1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3 = \left[\dfrac{n}{2}(n + 1)\right]^2.$

30. $1 + \dfrac{1}{\lfloor\underline{1}} + \dfrac{1}{\lfloor\underline{2}} + \dfrac{1}{\lfloor\underline{3}} + \cdots \text{ to } \infty = \lim_{n \doteq \infty}\left[1 + \dfrac{1}{n}\right]^n = e$
$$= 2.71828.$$

31. $e^x = 1 + x + \dfrac{x^2}{\lfloor\underline{2}} + \dfrac{x^3}{\lfloor\underline{3}} + \cdots \text{ to } \infty$ (for all values of x).

32. $\log_e(1 + x) = + x - \dfrac{x^2}{2} + \dfrac{x^3}{3} + \cdots \text{ to } \infty$ (if $-1 < x < +1$).

33. $e^{-x^2} = 1 - x^2 + \dfrac{x^4}{\lfloor\underline{2}} - \dfrac{x^6}{\lfloor\underline{3}} + \cdots \text{ to } \infty$ (for all values of x).

34. ${}_nP_r = n(n-1)(n-2) \cdots (n - r + 1) = \dfrac{\lfloor\underline{n}}{\lfloor\underline{n - r}}.$

35. ${}_nP_n = P_n = \lfloor\underline{n}.$

36. ${}_nP_r = n({}_{n-1}P_{r-1}).$

37. ${}_nC_r = \dfrac{1}{\lfloor\underline{r}}{}_nP_r = \dfrac{\lfloor\underline{n}}{\lfloor\underline{r}\ \lfloor\underline{n - r}}.$

Trigonometry

38. $\sin\alpha \csc\alpha = 1.$
39. $\cos\alpha \sec\alpha = 1.$
40. $\tan\alpha \cot\alpha = 1.$
41. $\sin^2\alpha + \cos^2\alpha = 1.$
42. $\sec^2\alpha = 1 + \tan^2\alpha.$
43. $\csc^2\alpha = 1 + \cot^2\alpha.$
44. $\dfrac{\sin\alpha}{\cos\alpha} = \tan\alpha.$

45. $\dfrac{\cos\alpha}{\sin\alpha} = \cot\alpha.$

46. $\sin(90° \pm \alpha) = \cos\alpha.$

47. $\cos(90° \pm \alpha) = \mp \sin\alpha.$

48. $\tan(90° \pm \alpha) = \mp \cot\alpha.$

49. $\sin(180° \pm \alpha) = \mp \sin\alpha.$

50. $\cos(180° \pm \alpha) = -\cos\alpha.$

51. $\tan(180° \pm \alpha) = \pm \tan\alpha.$

52. $\sin(270° \pm \alpha) = -\cos\alpha.$

53. $\cos(270° \pm \alpha) = \pm \sin\alpha.$

54. $\tan(270° \pm \alpha) = \mp \cot\alpha.$

55. $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta.$

56. $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta.$

57. $\tan(\alpha \pm \beta) = \dfrac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$

58. $\sin 2\alpha = 2\sin\alpha\cos\alpha.$

59. $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha.$

60. $= 1 - 2\sin^2\alpha.$

61. $= 2\cos^2\alpha - 1.$

62. $\tan 2\alpha = \dfrac{2\tan\alpha}{1 - \tan^2\alpha}.$

63. $\sin\dfrac{\alpha}{2} = \sqrt{\dfrac{1 - \cos\alpha}{2}}.$

64. $\cos\dfrac{\alpha}{2} = \sqrt{\dfrac{1 + \cos\alpha}{2}}.$

65. $\tan\dfrac{\alpha}{2} = \sqrt{\dfrac{1 - \cos\alpha}{1 + \cos\alpha}}$

66. $= \dfrac{1 - \cos\alpha}{\sin\alpha}$

67. $= \dfrac{\sin\alpha}{1 + \cos\alpha}.$

68. $\sin\alpha + \sin\beta = 2\sin\dfrac{\alpha+\beta}{2}\cos\dfrac{\alpha-\beta}{2}.$

69. $\sin\alpha - \sin\beta = 2\cos\dfrac{\alpha+\beta}{2}\sin\dfrac{\alpha-\beta}{2}.$

70. $\cos\alpha + \cos\beta = 2\cos\dfrac{\alpha+\beta}{2}\cos\dfrac{\alpha-\beta}{2}.$

71. $\cos\alpha - \cos\beta = -2\sin\dfrac{\alpha+\beta}{2}\sin\dfrac{\alpha-\beta}{2}.$

72. $\dfrac{a}{\sin\alpha} = \dfrac{b}{\sin\beta} = \dfrac{c}{\sin\gamma}.$ Law of sines.

73. $a^2 = b^2 + c^2 - 2bc \cos \alpha$. Law of cosines.

74. $\dfrac{\tan \dfrac{\alpha - \beta}{2}}{\tan \dfrac{\alpha + \beta}{2}} = \dfrac{a - b}{a + b}$. Law of tangents.

75. Area of triangle $= \sqrt{s(s-a)(s-b)(s-c)}$, where $2s = a + b + c$.

76. $= \dfrac{cb \sin \alpha}{2}$

77. $= \dfrac{c^2 \sin \alpha \sin \beta}{2 \sin(\alpha + \beta)}$.

78. Radius of inscribed circle $= \sqrt{\dfrac{(s-a)(s-b)(s-c)}{s}}$.

79. $\sin^2 \alpha = \frac{1}{2}(1 - \cos 2\alpha)$.

80. $\cos^2 \alpha = \frac{1}{2}(1 + \cos 2\alpha)$.

81. $\tan^2 \alpha = \dfrac{1 - \cos 2\alpha}{1 + \cos 2\alpha}$

82. $= \dfrac{(1 - \cos 2\alpha)^2}{\sin^2 2\alpha}$

83. $= \dfrac{\sin^2 2\alpha}{(1 + \cos 2\alpha)^2}$.

84. $\sin 3\alpha = 3 \sin \alpha - 4 \sin^3 \alpha$.

85. $\cos 3\alpha = 4 \cos^3 \alpha - 3 \cos \alpha$.

86. $\tan 3\alpha = \dfrac{3 \tan \alpha - \tan^3 \alpha}{1 - 3 \tan^2 \alpha}$.

87. $\alpha > \sin \alpha > \alpha - \dfrac{\alpha^3}{\underline{|3}}$ if α is expressed in radians.

88. $\alpha - \dfrac{\alpha^3}{\underline{|3}} < \sin \alpha < \alpha - \dfrac{\alpha^3}{\underline{|3}} + \dfrac{\alpha^5}{\underline{|5}}$ if α is expressed in radians.

If α is small and expressed in radians:

89. $\sin \alpha = \alpha$ approximately, or

90. $\sin \alpha = \alpha - \dfrac{\alpha^3}{\underline{|3}}$ is a better approximation, or

91. $\sin \alpha = \alpha - \dfrac{\alpha^3}{\underline{|3}} + \dfrac{\alpha^5}{\underline{|5}}$ is a still better approximation.

92. $1 > \cos \alpha > 1 - \dfrac{\alpha^2}{\underline{|2}}$ if α is expressed in radians.

93. $1 - \dfrac{\alpha^2}{\underline{|2}} < \cos \alpha < 1 - \dfrac{\alpha^2}{\underline{|2}} + \dfrac{\alpha^4}{\underline{|4}}$ if α is expressed in radians.

94. $1 - \frac{\alpha^2}{\lfloor 2} + \frac{\alpha^4}{\lfloor 4} > \cos\alpha > 1 - \frac{\alpha^2}{\lfloor 2} + \frac{\alpha^4}{\lfloor 4} - \frac{\alpha^6}{\lfloor 6}$ if α is expressed in radians.

If α is small and measured in radians:

95. $\cos\alpha = 1$ approximately, or

96. $\cos\alpha = 1 - \frac{\alpha^2}{\lfloor 2}$ is a better approximation, or

97. $\cos\alpha = 1 - \frac{\alpha^2}{\lfloor 2} + \frac{\alpha^4}{\lfloor 4}$ is a still better approximation.

Analytical Geometry

98. $y = ax + b$, slope equation of straight line.

99. $x\cos\alpha + y\sin\alpha = p$, normal equation of straight line.

100. $x^2 + y^2 = r^2$, equation of circle, center at origin, radius r.

101. $(x - \alpha)^2 + (y - \beta)^2 = r^2$, equation of circle, center at the point (α, β), radius r.

102. $y = px^2$, equation of parabola with axis coinciding with Y-axis.

103. $y = p(x - \alpha)^2$, equation of parabola with axis parallel to the Y-axis, but α units to its right (to its left if α is negative).

104. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, equation of ellipse, axes coinciding with the coordinate axes, foci upon X-axis.

105. Area of the ellipse $= \pi ab$.

106. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, equation of hyperbola, axes coinciding with coordinate axes.

107. $x^2 - y^2 = a^2$, equation of equilateral hyperbola, axes coinciding with coördinate axes.

108. $xy = k$, equation of equilateral hyperbola referred to asymptotes as axes.

109. $Ax + By + Cz = D$, equation of plane.

110. $x\cos\alpha + y\cos\beta + z\cos\gamma = p$, normal equation of plane.

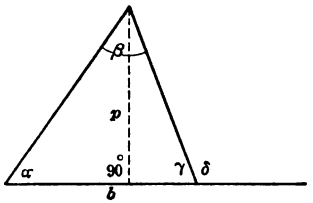
111. $x^2 + y^2 + z^2 = r^2$, equation of sphere, center at origin, radius r.

112. $(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$, equation of sphere, center at the point (a, b, c).

113. $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, where α, β, and γ are the direction angles of a line.

114. $\cos\theta = \cos\alpha_1 \cos\alpha_2 + \cos\beta_1 \cos\beta_2 + \cos\gamma_1 \cos\gamma_2$, where θ is the angle between two lines whose direction angles are, respectively, $\alpha_1, \beta_1, \gamma_1$, and $\alpha_2, \beta_2, \gamma_2$.

MENSURATION

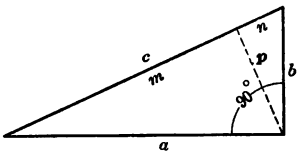


General Triangle

115. Area $= \frac{1}{2}bp$.

116. $\alpha + \beta + \gamma = 180°$.

117. $\delta = \alpha + \beta$.

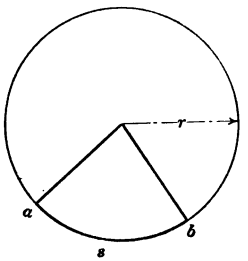


Right Triangle

118. $b^2 + a^2 = c^2$.

119. $p^2 = mn$.

120. $b^2 = cn$.



Circle

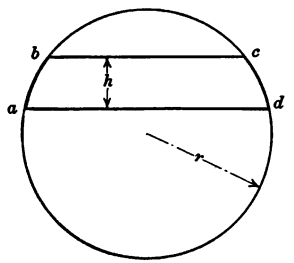
r = radius.

121. Circumference $= 2\pi r$.

122. Area $= \pi r^2$.

s = length of arc ab.

123. Area of sector $= \frac{1}{2}rs$.



Sphere

radius $= r$.

124. Volume $= \frac{4}{3}\pi r^3$.

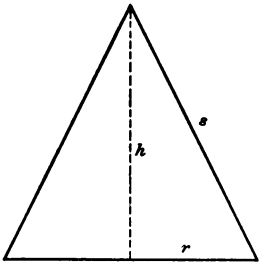
125. Area of surface $= 4\pi r^2$.

h = altitude of zone.

126. Area of zone $abcd = 2\pi rh$.

a, b = radii of bases of segment.

127. Volume of segment $= \frac{\pi h}{6}(3a^2 + 3b^2 + h^2)$.

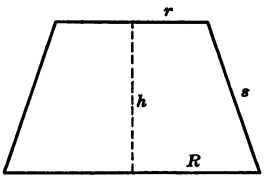


Right Circular Cone

r = radius of base.
h = altitude.
s = slant height.
$s = \sqrt{r^2 + h^2}$.

128. Volume $= \frac{1}{3}\pi r^2 h$.

129. Area of convex surface $= \pi r s$.



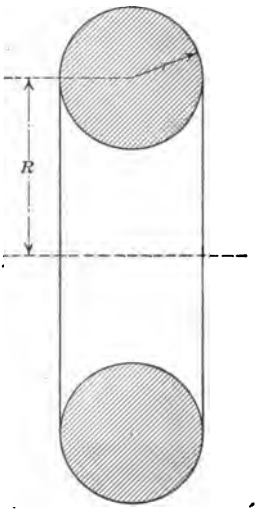
Frustum of Right Circular Cone

R = radius of lower base.
r = radius of upper base
h = altitude.
s = slant height.
$s = \sqrt{h^2 + (R - r)^2}$.

130. Area of convex surface $= \pi s(R - r)$.

131. Volume
$$= \frac{\pi h}{3}(R^2 + Rr + r^2).$$

132.
$$= \frac{\pi h(R^3 - r^3)}{3(R - r)}.$$



Anchor Ring

r = radius of generating circle.
R = mean radius.

133. Volume $= 2\pi^2 r^2 R$.

134. Area of surface $= 4\pi^2 r R$.

Theory of Probability

135. $y = \frac{h}{\sqrt{\pi}} e^{-h^2 x^2}$, probability curve.

136. $\sigma = \sqrt{\frac{\Sigma f v^2}{n}}$, standard deviation.

137. $E = \frac{0.6745}{n} \sqrt{\Sigma f v^2}$, probable error of mean.

TABLE XVI.—DECIMAL EQUIVALENTS OF COMMON FRACTIONS

8ths	16ths	8ths	16ths
1/8 = 0.125	1/16 = 0.0625	5/8 = 0.625	9/16 = 0.5625
1/4 = 0.250	3/16 = 0.1875	3/4 = 0.750	11/16 = 0.6875
3/8 = 0.375	5/16 = 0.3125	7/8 = 0.875	13/16 = 0.8125
1/2 = 0.500	7/16 = 0.4375		15/16 = 0.9375

TABLE XVII.—CONVERTING INCHES INTO FEET

Inches	0	1/8	1/4	3/8	1/2	5/8	3/4	7/8
0	0.000	0.010	0.021	0.031	0.042	0.052	0.062	0.073
1	0.083	0.094	0.104	0.115	0.125	0.135	0.146	0.156
2	0.167	0.177	0.188	0.198	0.208	0.219	0.229	0.239
3	0.250	0.260	0.271	0.281	0.292	0.302	0.312	0.323
4	0.333	0.344	0.354	0.365	0.375	0.385	0.396	0.406
5	0.417	0.427	0.437	0.448	0.458	0.469	0.479	0.490
6	0.500	0.510	0.521	0.531	0.542	0.552	0.562	0.573
7	0.583	0.594	0.604	0.615	0.625	0.635	0.646	0.656
8	0.667	0.677	0.688	0.698	0.708	0.719	0.729	0.740
9	0.750	0.760	0.771	0.781	0.792	0.802	0.812	0.823
10	0.833	0.844	0.854	0.865	0.875	0.885	0.896	0.906
11	0.917	0.927	0.937	0.948	0.958	0.969	0.979	0.990

TABLE XVIII.—BOARD MEASURE

Table giving the number of board feet in timbers of different dimensions; 1 board foot = 144 cubic inches.

Size in inches	Length in feet						
	8	10	12	14	16	18	20
1 × 4	2⅔	3⅓	4	4⅔	5⅓	6	6⅔
1 × 6	4	5	6	7	8	9	10
1 × 8	5⅓	6⅔	8	9⅓	10⅔	12	13⅓
1 × 10	6⅔	8⅓	10	11⅔	13⅓	15	16⅔
1 × 12	8	10	12	14	16	18	20
2 × 4	5⅓	6⅔	8	9⅓	10⅔	12	13⅓
2 × 6	8	10	12	14	16	18	20
2 × 8	10⅔	13⅓	16	18⅔	21⅓	24	26⅔
2 × 10	13⅓	16⅔	20	23⅓	26⅔	30	33⅓
2 × 12	16	20	24	28	32	36	40
3 × 4	8	10	12	14	16	18	20
3 × 6	12	15	18	21	24	27	30
3 × 8	16	20	24	28	32	36	40
3 × 10	20	25	30	35	40	45	50
3 × 12	24	30	36	42	48	54	60
4 × 4	10⅔	13⅓	16	18⅔	21⅓	24	26⅔
4 × 6	16	20	24	28	32	36	40
4 × 8	21⅓	26⅔	32	37⅓	42⅔	48	53⅓
4 × 10	26⅔	33⅓	40	46⅔	53⅓	60	66⅔
4 × 12	32	40	48	56	64	72	80
6 × 6	24	30	36	42	48	54	60
6 × 8	32	40	48	56	64	72	80
6 × 10	40	50	60	70	80	90	100
6 × 12	48	60	72	84	96	108	120
8 × 8	42⅔	53⅓	64	74⅔	85⅓	96	106⅔
8 × 10	53⅓	66⅔	80	93⅓	106⅔	120	133⅓
8 × 12	64	80	96	112	128	144	160
10 × 10	66⅔	83⅓	100	116⅔	133⅓	150	166⅔
10 × 12	80	100	120	140	160	180	200
12 × 12	96	120	144	168	192	216	240

TABLE XIX.—WATER CONTAINED IN 1 FOOT LENGTH OF CIRCULAR PIPE OF *d* INCHES INSIDE DIAMETER

d	Gallons	Weight lbs.	*d*	Gallons	Weight lbs.
½	0.010	0.085	7	1.999	16.683
1	0.041	0.340	8	2.611	21.790
1½	0.092	0.766	10	4.080	34.048
2	0.163	1.362	12	5.875	49.028
2½	0.255	2.128	14	7.997	66.733
3	0.367	3.064	16	10.44	87.162
3½	0.500	4.171	18	13.22	110.314
4	0.653	5.448	20	16.32	136.190
5	1.020	8.512	22	19.75	164.790
6	1.469	12.257	24	23.50	196.114

TABLE XX.—TABLE OF HEADS OF WATER CORRESPONDING TO GIVEN PRESSURE AND OF PRESSURE CORRESPONDING TO GIVEN HEAD

Pressure, lbs. per sq. in.	Head, feet	Head, feet	Pressure, lbs. per sq. in.
1	2.307	1	0.434
2	4.614	2	0.867
3	6.920	3	1.301
4	9.227	4	1.734
5	11.534	5	2.168
6	13.841	6	2.601
7	16.147	7	3.035
8	18.454	8	3.468
9	20.761	9	3.902
10	23.068	10	4.335
20	46.135	20	8.670
30	69.203	30	13.005
40	92.271	40	17.340
50	115.338	50	21.675
60	138.406	60	26.010
70	161.474	70	30.346
80	184.541	80	34.681
90	207.609	90	39.016
100	230.677	100	43.351

TABLE XXI.—CONVERSION TABLES

Lengths

1 centimeter	= 0.3937	inch.
1 meter	= 39.37	inches.
1 kilometer	= 0.62137	mile.
1 inch	= 2.540	centimeters.
1 foot	= 30.480	centimeters.
1 yard	= 0.9144	meter.
1 mile (5280 feet)	= 1.609	kilometers.

Areas

1 square centimeter	= 0.1550	square inch.
1 square meter	= 10.764	square feet.
1 square kilometer	= 0.3861	square mile.
1 square inch	= 6.4516	square centimeters.
1 square foot	= 0.0929	square meter.
1 square mile	= 2.590	square kilometers.

Volume

1 cubic centimeter	= 0.0610	cubic inch.
1 cubic meter	= 1.308	cubic yards.
1 liter	= 61.023	cubic inches = 1.057 quarts (liquid).
1 cubic inch	= 16.387	cubic centimeters.
1 cubic foot	= 7.480	gallons (liquid) = 28.317 liters.
1 cubic yard	= 0.7646	cubic meter.
1 pint (liquid)	= 28.875	cubic inches = 0.4732 liter.
1 gallon	= 231	cubic inches = 3.785 liters.

Weights

1 gram	= 15.432	grains = 0.03527 ounce (Av.).
1 kilogram	= 2.2046	pounds (Av.).
1 grain	= 0.0022857	ounce (Av.) = 0.064799 gram.
1 ounce (Av.)	= 437.5	grains = 28.3495 grams.
1 pound (Av.)	= 0.45359	kilogram = 7000 grains.

Pressure

1 foot of water column	= 0.4335	pounds per square inch.
	= 22.419	millimeters of mercury.
	= 0.8826	inch of mercury.
	= 0.0295	atmosphere.
1 inch of mercury column	= 1.133	feet of water.
	= 0.4912	pound per square inch.
	= 0.03342	atmosphere.
1 pound per square inch	= 51.712	millimeters of mercury.
	= 2.307	feet of water.
	= 2.036	inches of mercury.
1 atmosphere	= 760	millimeters of mercury.
	= 33.901	feet of water.
	= 29.921	inches of mercury.
	= 14.697	pounds per square inch.

Weight and Volume of Water

1 cubic centimeter water	= 1	gram.
1 pint (liquid) water	= 1.043	pounds.
1 cubic foot water	= 62.428	pounds (Av.).
1 ounce (Av.)	= 28.35	cubic centimeters water.
	= 1.730	cubic inches water.

Flow of Water

1 cubic foot per second	= 450	gallons per minute (approximately).
1 acre foot per day	= 0.5042	cubic foot per second.

Salts in Solution

1 part per million	= 0.05842	grain per gallon.
1 grain per gallon	= 17.118	parts per million.

TABLE XXII.—NATURAL TRIGONOMETRIC FUNCTIONS

°	′	N.Sin	N.Tan	N.Cot	N.Cos		
0	0	0.0000	0.0000	∞	1.0000	0	90
	10	029	029	343.77	000	50	
	20	058	058	171.89	000	40	
	30	087	087	114.59	000	30	
	40	116	116	85.940	0.9999	20	
	50	145	146	68.750	999	10	
1	0	0.0175	0.0175	57.290	0.9998	0	89
	10	204	204	49.104	998	50	
	20	233	233	42.964	997	40	
	30	262	262	38.189	997	30	
	40	291	291	34.368	996	20	
	50	320	320	31.242	995	10	
2	0	0.0349	0.0349	28.636	0.9994	0	88
	10	378	378	26.432	993	50	
	20	407	407	24.542	992	40	
	30	436	437	22.904	990	30	
	40	465	466	21.470	989	20	
	50	494	495	20.206	988	10	
3	0	0.0523	0.0524	19.081	0.9986	0	87
	10	552	553	18.075	985	50	
	20	581	582	17.169	983	40	
	30	610	612	16.350	981	30	
	40	640	641	15.605	980	20	
	50	669	670	14.924	978	10	
4	0	0.0698	0.0699	14.301	0.9976	0	86
	10	727	729	13.727	974	50	
	20	756	758	13.197	971	40	
	30	785	787	12.706	969	30	
	40	814	816	12.251	967	20	
	50	843	846	11.826	964	10	
5	0	0.0872	0.0875	11.430	0.9962	0	85

°	′	N.Sin	N.Tan	N.Cot	N.Cos		
5	0	0.0872	0.0875	11.430	0.9962	0	85
	10	901	904	11.059	959	50	
	20	929	934	10.712	957	40	
	30	958	963	10.385	954	30	
	40	987	992	10.078	951	20	
	50	0.1016	0.1022	9.7882	948	10	
6	0	0.1045	0.1051	9.5144	0.9945	0	84
	10	074	080	2553	942	50	
	20	103	110	0098	939	40	
	30	132	139	8.7769	936	30	
	40	161	169	5555	932	20	
	50	190	198	3450	929	10	
7	0	0.1219	0.1228	8.1443	0.9925	0	83
	10	248	257	7.9530	922	50	
	20	276	287	7704	918	40	
	30	305	317	5958	914	30	
	40	334	346	4287	911	20	
	50	363	376	2687	907	10	
8	0	0.1392	0.1405	7.1154	0.9903	0	82
	10	421	435	6.9682	899	50	
	20	449	465	8269	894	40	
	30	478	495	6912	890	30	
	40	507	524	5606	886	20	
	50	536	554	4348	881	10	
9	0	0.1564	0.1584	6.3138	0.9877	0	81
	10	593	614	1970	872	50	
	20	622	644	0844	868	40	
	30	650	673	5.9758	863	30	
	40	679	703	8708	858	20	
	50	708	733	7694	853	10	
10	0	0.1736	0.1763	5.6713	0.9848	0	80

°	′	N.Sin	N.Tan	N.Cot	N.Cos		
10	0	0.1736	0.1763	5.6713	0.9848	0	80
	10	765	793	5764	843	50	
	20	794	823	4845	838	40	
	30	822	853	3955	833	30	
	40	851	883	3093	827	20	
	50	880	914	2257	822	10	
11	0	0.1908	0.1944	5.1446	0.9816	0	79
	10	937	974	0658	811	50	
	20	965	0.2004	4.9894	805	40	
	30	994	035	9152	799	30	
	40	0.2022	065	8430	793	20	
	50	051	095	7729	787	10	
12	0	0.2079	0.2126	4.7046	0.9781	0	78
	10	108	156	6382	775	50	
	20	136	186	5736	769	40	
	30	164	217	5107	763	30	
	40	193	247	4494	757	20	
	50	221	278	3897	750	10	
13	0	0.2250	0.2309	4.3315	0.9744	0	77
	10	278	339	2747	737	50	
	20	306	370	2193	730	40	
	30	334	401	1653	724	30	
	40	363	432	1126	717	20	
	50	391	462	0611	710	10	
14	0	0.2419	0.2493	4.0108	0.9703	0	76
	10	447	524	3.9617	696	50	
	20	476	555	9136	689	40	
	30	504	586	8667	681	30	
	40	532	617	8208	674	20	
	50	560	648	7760	667	10	
15	0	0.2588	0.2679	3.7321	0.9659	0	75

	N.Cos	N.Cot	N.Tan	N.Sin	′	°

TABLE XXII.—(*Continued*)

° ′	N.Sin	N.Tan	N.Cot	N.Cos		° ′	N.Sin	N.Tan	N.Cot	N.Cos		° ′	N.Sin	N.Tan	N.Cot	N.Cos	
15 0	0.2588	0.2679	3.7321	0.9659	0 **75**	**20** 0	0.3420	0.3640	2.7475	0.9397	0 **70**	**25** 0	0.4226	0.4663	2.1445	0.9063	0 **65**
10	616	711	6891	652	50	10	448	673	228	387	50	10	253	699	283	051	50
20	644	742	6470	644	40	20	475	706	2.6985	377	40	20	279	734	123	038	40
30	672	773	6059	636	30	30	502	739	746	367	30	30	305	770	2.0965	026	30
40	700	805	5656	628	20	40	529	772	511	356	20	40	331	806	809	013	20
50	728	836	5261	621	10	50	557	805	279	346	10	50	358	841	655	001	10
16 0	0.2756	0.2867	3.4874	0.9613	0 **74**	**21** 0	0.3584	0.3839	2.6051	0.9336	0 **69**	**26** 0	0.4384	0.4877	2.0503	0.8988	0 **64**
10	784	899	4495	605	50	10	611	872	2.5826	325	50	10	410	913	353	975	50
20	812	931	4124	596	40	20	638	906	605	315	40	20	436	950	204	962	40
30	840	962	3759	588	30	30	665	939	386	304	30	30	462	986	057	949	30
40	868	994	3402	580	20	40	692	973	172	293	20	40	488	0.5022	1.9912	936	20
50	896	0.3026	3052	572	10	50	719	0.4006	2.4960	283	10	50	514	059	768	923	10
17 0	0.2924	0.3057	3.2709	0.9563	0 **73**	**22** 0	0.3746	0.4040	2.4751	0.9272	0 **68**	**27** 0	0.4540	0.5095	1.9626	0.8910	0 **63**
10	952	089	2371	555	50	10	773	074	545	261	50	10	566	132	486	897	50
20	979	121	2041	546	40	20	800	108	342	250	40	20	592	169	347	884	40
30	0.3007	153	1716	537	30	30	827	142	142	239	30	30	617	206	210	870	30
40	035	185	1397	528	20	40	854	176	2.3945	228	20	40	643	243	074	857	20
50	062	217	1084	520	10	50	881	210	750	216	10	50	669	280	1.8940	843	10
18 0	0.3090	0.3249	3.0777	0.9511	0 **72**	**23** 0	0.3907	0.4245	2.3559	0.9205	0 **67**	**28** 0	0.4695	0.5317	1.8807	0.8829	0 **62**
10	118	281	0475	502	50	10	934	279	369	194	50	10	720	354	676	816	50
20	145	314	0178	492	40	20	961	314	183	182	40	20	746	392	546	802	40
30	173	346	2.9887	483	30	30	987	348	2.2998	171	30	30	772	430	418	788	30
40	201	378	9600	474	20	40	0.4014	383	817	159	20	40	797	467	291	774	20
50	228	411	9319	465	10	50	041	417	637	147	10	50	823	505	165	760	10
19 0	0.3256	0.3443	2.9042	0.9455	0 **71**	**24** 0	0.4067	0.4452	2.2460	0.9135	0 **66**	**29** 0	0.4848	0.5543	1.8040	0.8746	0 **61**
10	283	476	8770	446	50	10	094	487	286	124	50	10	874	581	1.7917	732	50
20	311	508	8502	436	40	20	120	522	113	112	40	20	899	619	796	718	40
30	338	541	8239	426	30	30	147	557	2.1943	100	30	30	924	658	675	704	30
40	365	574	7980	417	20	40	173	592	775	088	20	40	950	696	556	689	20
50	393	607	7725	407	10	50	200	628	609	075	10	50	975	735	437	675	10
20 0	0.3420	0.3640	2.7475	0.9397	0 **70**	**25** 0	0.4226	0.4663	2.1445	0.9063	0 **65**	**30** 0	0.5000	0.5774	1.7321	0.8660	0 **60**
	N.Cos	N.Cot	N.Tan	N.Sin	′ °		N.Cos	N.Cot	N.Tan	N.Sin	′ °		N.Cos	N.Cot	N.Tan	N.Sin	′ °

TABLE XXII.—(*Continued*)

°	′	N.Sin	N.Tan	N.Cot	N.Cos		°	′	N.Sin	N.Tan	N.Cot	N.Cos		°	′	N.Sin	N.Tan	N.Cot	N.Cos	
30	0	0.5000	0.5774	1.7321	0.8660	0 **60**	**35**	0	0.5736	0.7002	1.4281	0.8192	0 **55**	**40**	0	0.6428	0.8391	1.1918	0.7660	0 **50**
	10	025	812	205	646	50		10	760	046	193	175	50		10	450	441	847	642	50
	20	050	851	090	631	40		20	783	089	106	158	40		20	472	491	778	623	40
	30	075	890	1.6977	616	30		30	807	133	019	141	30		30	494	541	708	604	30
	40	100	930	864	601	20		40	831	177	1.3934	124	20		40	517	591	640	585	20
	50	125	969	753	587	10		50	854	221	848	107	10		50	539	642	571	566	10
31	0	0.5150	0.6009	1.6643	0.8572	0 **59**	**36**	0	0.5878	0.7265	1.3764	0.8090	0 **54**	**41**	0	0.6561	0.8693	1.1504	0.7547	0 **49**
	10	175	048	534	557	50		10	901	310	680	073	50		10	583	744	436	528	50
	20	200	088	426	542	40		20	925	355	597	056	40		20	604	796	369	509	40
	30	225	128	319	526	30		30	948	400	514	039	30		30	626	847	303	490	30
	40	250	168	212	511	20		40	972	445	432	021	20		40	648	899	237	470	20
	50	275	208	107	496	10		50	995	490	351	004	10		50	670	952	171	451	10
32	0	0.5299	0.6249	1.6003	0.8480	0 **58**	**37**	0	0.6018	0.7536	1.3270	0.7986	0 **53**	**42**	0	0.6691	0.9004	1.1106	0.7431	0 **48**
	10	324	289	1.5900	465	50		10	041	581	190	969	50		10	713	057	041	412	50
	20	348	330	798	450	40		20	065	627	111	951	40		20	734	110	1.0977	392	40
	30	373	371	697	434	30		30	088	673	032	934	30		30	756	163	913	373	30
	40	398	412	597	418	20		40	111	720	1.2954	916	20		40	777	217	850	353	20
	50	422	453	497	403	10		50	134	766	876	898	10		50	799	271	786	333	10
33	0	0.5446	0.6494	1.5399	0.8387	0 **57**	**38**	0	0.6157	0.7813	1.2799	0.7880	0 **52**	**43**	0	0.6820	0.9325	1.0724	0.7314	0 **47**
	10	471	536	301	371	50		10	180	860	723	862	50		10	841	380	661	294	50
	20	495	577	204	355	40		20	202	907	647	844	40		20	862	435	599	274	40
	30	519	619	108	339	30		30	225	954	572	826	30		30	884	490	538	254	30
	40	544	661	013	323	20		40	248	0.8002	497	808	20		40	905	545	477	234	20
	50	568	703	1.4919	307	10		50	271	050	423	790	10		50	926	601	416	214	10
34	0	0.5592	0.6745	1.4826	0.8290	0 **56**	**39**	0	0.6293	0.8098	1.2349	0.7771	0 **51**	**44**	0	0.6947	0.9657	1.0355	0.7193	0 **46**
	10	616	787	733	274	50		10	316	146	276	753	50		10	967	713	295	173	50
	20	640	830	641	258	40		20	338	195	203	735	40		20	988	770	235	153	40
	30	664	873	550	241	30		30	361	243	131	716	30		30	0.7009	827	176	133	30
	40	688	916	460	225	20		40	383	292	059	698	20		40	030	884	117	112	20
	50	712	959	370	208	10		50	406	342	1.1988	679	10		50	050	942	058	092	10
35	0	0.5736	0.7002	1.4281	0.8192	0 **55**	**40**	0	0.6428	0.8391	1.1918	0.7660	0 **50**	**45**	0	0.7071	1.0000	1.0000	0.7071	0 **45**
		N.Cos	N.Cot	N.Tan	N.Sin	′ °			N.Cos	N.Cot	N.Tan	N.Sin	′ °			N.Cos	N.Cot	N.Tan	N.Sin	′ °

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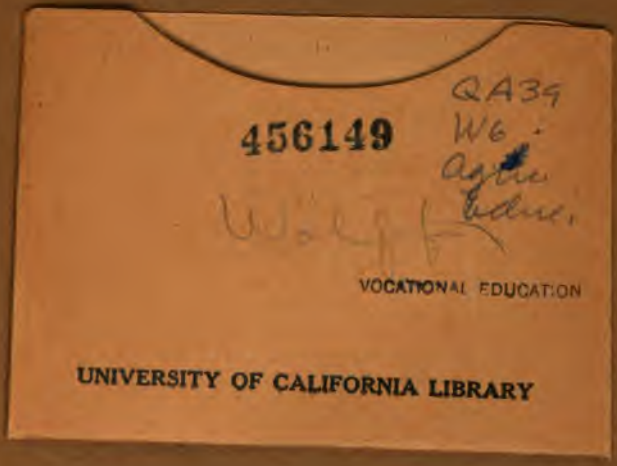
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